EULER'S CONSTANT AND SERIES TRANSFORMATIONS WITH BASES OF BALANCED LATTICES

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Abstract. The distance of the numbers

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m-1}-\log m$$

from the Euler-Mascheroni constant γ was expressed by J. Ser in 1925 using a rapidly converging combinatorial series of rational numbers. We provide a series transformation of Ser's formula using the basis B of a lattice to eliminate the transcendental logarithm values. For this purpose, we need small solutions of systems of linear forms and use an idea by Becker-Landeck from 1988 for reducing the system of linear forms to a homogeneous system of equations. From this series transformation of Ser's formula, rational approximations for γ could be constructed which allow to prove the irrationality of Euler's constant γ, provided that the lengths of the vectors from the basis B (with respect to the maximum norm) do not differ too much from each other.

2020 *Mathematics Subject Classification.* Primary 11J72, 52C07. Secondary 40A05, 65B10.

Key words and phrases. Euler's constant, irrationality, series transformation, generalization of Siegel's lemma, basis reduction, LLL algorithm, $\mathbb{Z}$ modules.

Amamus detegere arcana numerorum.
(Chr.R.Havens and C.Elsner, 2025)

1. A NEW STRATEGY TO ATTACK AN OLD PROBLEM

1.1. The general principle.

To prove the irrationality of a given (positive) real number ξ, the following procedure is often used: Two strictly monotonically increasing sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ of natural numbers are explicitly constructed so that the sequence $(|b_n\xi - a_n|)_{n\geqslant 0}$ converges strictly monotonically to zero as n increases. In some cases, the numbers a_n and b_n are derived from the partial sums of a series converging to ξ, whose summands are positive rational numbers. A classic example of this method is given by the proof of irrationality for the number $e = \exp(1)$, which is based on the Taylor series of the exponential function at the point 1:

From this Taylor series we have the following inequalities,

$$\frac{1}{n+1} < q_n e - p_n < \frac{1}{n}\,, \tag{1.1.1}$$

where $q_n := n!$ and $p_n := \lfloor n!e \rfloor$. We want to express n in terms of q_n. For this purpose, we apply the inequalities

$$e^n < e^{n+1} < n! < n^n \qquad (n \geqslant 7)\,.$$

It turns out that

$$\log(n!) < n\log n < n\log\log(n!) \qquad \text{and} \qquad n+1 < \log(n!) \qquad (n \geqslant 7)\,,$$

and therefore we have

$$n > \frac{\log(n!)}{\log\log(n!)} = \frac{\log q_n}{\log\log q_n} \qquad \text{and} \qquad n+1 < \log q_n \qquad (n \geqslant 7)\,.$$

We etimate both sides of the inequalities in (1.1.1) by

$$\frac{1}{\log q_n} < q_n e - p_n < \frac{\log\log q_n}{\log q_n} \qquad (n \geqslant 7)\,.$$

For increasing q_n, the term

$$\frac{\log\log q_n}{\log q_n}$$

tends very slowly to zero. But this argument is sufficient to prove the irrationality of e. It is worse to mention that the rationals p_n/q_n are far away to be identified as convergents obtained from the continued fraction expansion of e .

However, many series of rational numbers are not suitable for proving the irrationality of their limit by this method, because their partial sums a_n/b_n have a denominator b_n that is too large compared to the convergence rate of the series. In these cases, the sequence $(|b_n\xi - a_n|)_{n\geqslant 0}$ would no longer converge to zero and the method fails. In order to make a series accessible for an irrationality proof of Euler's constant γ, the author proposes a special series transformation that is performed with the basis vectors of a so-called norm-balanced lattice. These lattices are formed using the partial sums of a series itself. We will first present the construction of these bases here in a special definition and then discuss its applications:

The basic relation between a set of rationals and a balanced lattice.

Let $\{q_0, q_1, \dots, q_N\}$ be a set of positive rational numbers, say,

$$q_\nu = \frac{r_\nu}{s_\nu} \qquad (\nu = 0, 1, \dots, N)$$

with coprime positive integers r_ν and s_ν. Set

$$g := \operatorname{lcm}\bigl(s_0, s_1, \dots, s_N\bigr)$$

and

$$\vec{b}_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ g \end{pmatrix}, \quad \vec{b}_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ gq_1 \end{pmatrix}, \quad \dots \quad, \vec{b}_{N-1} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ gq_{N-1} \end{pmatrix}, \quad \vec{b}_N := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ gq_N \end{pmatrix}.$$

Here, we have $N+1$ linear independent integer vectors of dimension $N+2$. We consider the $(N+1)$ dimensional lattice

$$\Lambda := \operatorname{span}_{\mathbb{Z}}\Bigl(\vec{b}_0, \vec{b}_1, \dots, \vec{b}_N\Bigr),$$

where the lattice constant $d(\Lambda)$ is given by

$$d(\Lambda) = \sqrt{1 + g^2 + (gq_1)^2 + (gq_2)^2 + \dots + (gq_N)^2}\,.$$

Moreover, we have the scalar product

$$\vec{b}_\nu \cdot \bigl(g, gq_1, gq_2, \dots, gq_N, -1\bigr) = 0 \qquad (0 \leqslant \nu \leqslant N)\,.$$

Next, let $\{\vec{t}_1, \vec{t}_2, \dots, \vec{t}_{N+1}\}$ be a basis of an $N+1$ dimensional sublattice Λ_{sub} of Λ, where the vectors $\vec{t}_\nu$ are indexed such that

$$T_1 \leqslant T_2 \leqslant \dots \leqslant T_{N+1} \qquad \text{with the maximum norm} \qquad T_\nu := \|\vec{t}_\nu\|_\infty$$

for $\nu = 1, 2, \dots, N+1$. By a theorem of *Bombieri and Vaaler* [3], or by the second main theorem of *Minkowski* on lattices, there exists a basis $\{\vec{t}_1, \vec{t}_2, \dots, \vec{t}_{N+1}\}$ of Λ satisfying

$$T_1 T_2 \cdot \ldots \cdot T_{N+1} \leqslant d(\Lambda)\,.$$

Definition 1. *Let $\varepsilon \geqslant 0$. A set $\{q_0, q_1, \dots, q_N\}$ of positive rational numbers is said to generate an ε-balanced lattice of dimension $N+1$, if there is a basis $\{\vec{t}_1, \vec{t}_2, \dots, \vec{t}_{N+1}\}$ of an $N+1$ dimensional sublattice Λ_{sub} of Λ such that the following inequalities hold simultaneously:*

$$T_1 T_2 \cdot \ldots \cdot T_{N+1} \leqslant N^{\varepsilon N} d(\Lambda)\,, \tag{1.1.2}$$

$$T_1 \leqslant \sqrt{T_{N+1}}\,. \tag{1.1.3}$$

While inequality (1.1.2) is always easy to satisfy (even with $\varepsilon = 0$) based on the above-mentioned knowledge from lattice theory, the core of Definition 1 lies in the second inequality (1.1.3). It shows that the maximum norms of the basis vectors $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{N+1}$ do not differ too much from each other. This is expressed by designating the base as *balanced*.

Because the vectors $\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_N$ already form a basis of the lattice Λ and generally have a very large last component, the existence of a basis with smaller vectors in terms of the maximum norm is primarily a question of the theory of basis reduction for lattices. In the present case, however, the author has not yet found an answer to the question of which additional properties of the rational numbers $q_0, q_1, \ldots, q_N$ in Definition 1 guarantee a basis with the properties (1.1.2) and (1.1.3) for an $N+1$-dimensional sublattice Λ_{sub} of Λ. Therefore, the statement about the irrationality of γ for which we have applied this principle remains only a conjecture in this paper. However, in sample calculations with the LLL algorithm in order to reduce bases $\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_N$, generated by rational partial sums in the context of γ, we obtain 0-related bases of the original (balanced) lattice $\Lambda (= \Lambda_{sub})$: see the three examples in the Appendix at the end of this paper in Section 4.5.

Euler's constant γ is the limit of a certain series that converges exponentially with an error term C_1^{-n}. However, the partial sums of this series can only be expressed by fractions whose denominators are majored by terms of the form $C_2^n n!$, even with $C_2 > C_1$. Thus, an approach to proving irrationality of γ as above for the number e is out of the question. But the situation changes fundamentally after a series transformation with bases of suitable lattices. In doing so, the series transformation also eliminates transcendental logarithmic values, which are not needed at all for the argumentation. However, we have to struggle with an increasing formalism caused by the additional logarithm elimination process.

The starting point is Ser's formula, which has been already used by P. Appell in a failed attempt to prove the irrationality of γ in 1926. In the present proof attempt, numerous tools from combinatorial number theory, linear algebra, lattice theory, and Diophantine approximation are used. Overall, however, the argumentation is grouped around three main lemmas, which form the fundamental structure of our approach. The third main lemma, which is still a conjecture, is the statement about an ε-balanced lattice generated by certain rational partial sums. If this lemma is true, then γ is an irrational number.

This voluminous paper does not contain any theorems by the author, because the main statement depends on a conjecture about lattices with bases balanced in length. In addition, calculations and arguments are presented in a level of detail that is unusual for journal articles. For this reason, this work was not submitted in this form to a renowned professional journal. It appears here in *arXiv*, motivated by the hope that someone will find a proof for the outstanding conjecture about the ε-balanced bases. In this case, the author would be very interested in collaborating and would appreciate being contacted!

1.2. **The structure of the paper.**

In proofs of transcendence or proofs of the algebraic independence of a set of numbers, Siegel's lemma (or a variant of it) is often used, [5, Lemma 3, p. 106]. It guarantees nontrivial small solutions of a homogeneous linear system of equations, which can no longer be given constructively in closed form. For the approach to the Euler-Mascheroni constant

$$\gamma = \lim_{m\to\infty}\Big(\sum_{j=1}^{m-1}\frac{1}{j} - \log m\Big), \tag{1.2.1}$$

we will again use the method described in the previous Section 1.1. However, we will derive the (possible) existence of a sequence of γ-approximating fractions $(a_n/b_n)_{n\geqslant 1}$ without knowing explicitly the integers a_n and b_n. The method used here goes back to a paper [6] of the author from 1989, which is entitled *Über den Versuch eines nichtkonstruktiven Irrationalitätsbeweises für die Eulersche Konstante*[1] to indicate that the sequence $(a_n/b_n)_{n\geq 1}$ is not given constructively. In that work, a conjecture is made about a linear, Diophantine system of linear forms and was applied in such a way that the proof of the irrationality of Euler's constant remained incomplete. In the present paper, our system consists of M linear Diophantine equations $L_\mu(x_0,\dots,x_N) = \sum_{\nu=0}^{N} a_{\mu\nu}x_\nu$ $(\mu = 1,\dots,M)$ and one more linear equation $L_{M+1}(x_0,\dots,x_N) = \sum_{\nu=0}^{N} b_\nu x_\nu$, which is linearly independent over $\mathbb{Q}$ from the forms $L_1,\dots,L_M$. In this system, described by a $(M+1)\times(N+1)$ matrix[2] $\mathcal{A}$, there are fewer linear forms than unknowns. The number M of equations is estimated using the prime number theorem[3]. The coefficients of the form L_1 are greater than $n!$, while the coefficients of the remaining forms are bounded[4] by $2\log n$, because these forms are used to eliminate the logarithms from the original sequence. This difference in size is utilized significantly in the further proof.

Ultimately, the variables $x_0,\dots,x_N$ should take small integer values so that

$$\begin{aligned} L_\mu(x_0,\dots,x_N) &= 0 \qquad (\mu = 1,\dots,M),\\ L_{M+1}(x_0,\dots,x_N) &\neq 0 \end{aligned}$$

is considered as a solution of the system of linear forms[5]. Using a regular square matrix $\mathcal{C}$, this system is first converted into an equivalent system, described[6] by $\mathcal{AC}$, in such a way that the size of the coefficients of the new system is reduced and the integer elements of $\mathcal{C}$ are not too large[7]. When transforming the original system, very sharp estimates must be carried out around the transformation matrix[8] $\mathcal{C}$. For a small solution of the transformed system[9], we use an idea from a paper of P.G. Becker - Landeck [2]: The system is reduced to a quadratic system which can be solved with small integers by a theorem of Bombieri and Vaaler [3] from a paper published in 1983.

The construction of the $(N+1)\times(N+1)$ transformation matrix $\mathcal{C}$ forms the key aspect of

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[1] *On an attempt to prove the irrationality of Euler's constant in a non-constructive manner.* The paper [6] is a preprint published to stimulate research within the Mathematical Institute of the University of Hannover, Germany.

[2] (2.3.12)

[3] (2.4.48)

[4] (2.4.39) and (2.4.41) in Lemma 12

[5] (2.3.1), (2.3.2) and (2.3.9), (2.3.10)

[6] (2.3.14)

[7] Lemma 8 and (3.1.21)

[8] (3.1.21) and (3.2.4) to (3.2.8)

[9] (2.3.23) and Lemma 11

the proof[10]. It is formed using the vectors of a lattice base of the linear Diophantine equation

$$a_{1,0}x_0 + a_{1,1}x_1 + \ldots + a_{1,N}x_N - x_{N+1} \;=\; 0$$

where the coefficients $a_{1,0}, \ldots, a_{1,N}$ are taken from the linear form L_1 (see (2.5.5)). An integer lattice base of the solution space of this equation can easily be specified. However, in order to guarantee the required properties of the transformation matrix $\mathcal{C}$, this base must be reduced. The reduction process results in a small base, but at the same time it does not allow any of the base vectors to dominate significantly on the lengths of the other base vectors with respect to the maximum norm. Here, a set of rational numbers taken from the partial sums of a series generates an ε-balanced lattice: see Definition 1. From these rational numbers, we also generate the coefficients of the linear form L_1.

The starting point for our investigations are rapidly convergent combinatorial sequences for the error terms on the right-hand side of the formula

$$\gamma - \Big(\sum_{j=1}^{m-1} \frac{1}{j} - \log m \Big) \;=\; \frac{1}{n} \sum_{m=0}^{\infty} \frac{C_{m+1}}{(m+1)!\binom{m+n}{m}} \qquad (m, n \in \mathbb{N})\,. \tag{1.2.2}$$

E.P.Adams gave this formula already in 1922, independently of J.Ser, who mentioned it 1925 in [14]. However, not one of these authors provides a proof for (1.2.2). The formula is often cited in the literature, but nowhere is it completely proven. In [10], S. Krämer proved in parts the formula (1.2.2) in 2005. For the sake of completeness and for the convenience of our readers, we provide a complete proof of this long-known formula as well as for the integral representation[11] of the numbers C_m, the so-called Bernoulli coefficients of the second kind. We largely follow the approach proposed by S.Krämer in [10], although we could also take a completely different direction. Our approach uses only three known integral formulas (2.1.1) to (2.1.3) and three summation formulas involving binomial coefficients in Lemma 2.

Then, we start to develop the theory for the series transformation mentioned above to eliminate the transcendental logarithm values. Then, we try to prove the existence of a sequence $(a_n/b_n)_{n\geqslant 1}$ of γ-approximating fractions. Finally, we state the following conjecture.

Conjecture 1. *The method of series transformation, applied to the formula in (1.2.2), guarantees the existence of infinitely many integers a_n and b_n satisfying*

$$0 \;<\; \big|\, b_n\gamma - a_n \,\big| \;<\; 0.54^n \;<\; \frac{1}{b_n^{\delta}}\,, \tag{1.2.3}$$

where

$$\delta \;=\; \frac{\log 2 - 3\log 3 + 2\log 5}{\log 3 - 2\log 5 + \log 23} \;=\; 0.606942\ldots\,. \tag{1.2.4}$$

More than a dozen parameters are used for the method of the series transformation. Some are constant quantities, others depend on the main parameter n (like N and M) which is the subscript of the integers a_n and b_n in (1.2.3). This main parameter must be chosen very large, at least we need

$$n \;\geqslant\; 2^{27\,411\,206}\,.$$

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[10] Section 3.2

[11] Section 2.1

The structure of the whole paper is best described by listing the headings of the subsequent individual chapters and sections:

Because many incorrect or incomplete attempts have been made over the years to prove the irrationality of Euler's constant by some authors (for example, see [1]), we will carefully proceed with our arguments in an unusually detailed manner.

2. Summary of tools.

2.1. **The formula of J.Ser and its proof.**

Throughout this paper, log(.) denotes the natural logarithm. We start with three well-known integral formulas. Here, let $k \geqslant 2$ be a natural number.

$$
\begin{aligned}
\gamma &= \int_0^1 \Big(\frac{1}{\log t} + \frac{1}{1-t} \Big)\, dt\,, && (2.1.1)\\
\log k &= \int_0^1 \frac{t^{k-1}-1}{\log t}\, dt\,, && (2.1.2)\\
H_{k-1} &= 1 + \frac{1}{2} + \ldots + \frac{1}{k-1} \;=\; \int_0^1 \big(1 + t + t^2 + \ldots + t^{k-2} \big)\, dt\\
&= \int_0^1 \frac{1-t^{k-1}}{1-t}\, dt\,. && (2.1.3)
\end{aligned}
$$

See [7, 4.281, no.1] for (2.1.1) and [7, 4.267, no.8] for (2.1.2). Then, we obtain

$$
\gamma + \log n - H_{k-1} \;=\; \int_0^1 t^{k-1} \Big(\frac{1}{\log t} + \frac{1}{1-t} \Big)\, dt \;=\; \int_0^1 (1-t)^{k-1} \Big(\frac{1}{\log(1-t)} + \frac{1}{t} \Big)\, dt\,, \qquad (2.1.4)
$$

where t is replaced by $1-t$. Next, let $t \in \mathbb{R}$ with $|t| < 1$. The Bernoulli numbers C_m of the second kind are defined by the Taylor expansion of the function

$$
f(t) \;:=\; \frac{t}{\log(1-t)} \;=\; \sum_{m=0}^{\infty} \frac{C_m}{m!} t^m \;=\; -1 + \sum_{m=1}^{\infty} \frac{C_m}{m!} t^m\,;
$$

note that $\lim_{t\to 0} f(z) = -1$. Dividing by t yields

$$
\frac{1}{\log(1-t)} + \frac{1}{t} \;=\; \sum_{m=1}^{\infty} \frac{C_m}{m!} t^{m-1}\,. \qquad (2.1.5)
$$

The last tool we need before combining the results obtained so far is the beta function

$$
B(x,y) \;=\; \int_0^1 t^{x-1}(1-t)^{y-1}\, dt\,,
$$

which can be expressed by factorials for positive integers $x = m$ and $y = k$:

$$
B(m,k) \;=\; \frac{(n-1)!(m-1)!}{(k+m-1)!} \;=\; \frac{1}{k\binom{m+k-1}{m-1}}\,. \qquad (2.1.6)
$$

If we substitute (2.1.5) into (2.1.4) and interchange the integral with the sum, we obtain

$$
\begin{aligned}
\gamma + \log k - H_{k-1} &= \lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} (1-t)^{k-1} \Big(\frac{1}{\log(1-t)} + \frac{1}{t} \Big)\, dt\\
&= \lim_{\varepsilon\to 0^+} \sum_{m=1}^{\infty} \frac{C_m}{m!} \int_0^{1-\varepsilon} t^{m-1}(1-t)^{k-1}\, dt \;=\; \sum_{m=1}^{\infty} \frac{C_m}{m!} \cdot B(m,k)\\
&\overset{(2.1.6)}{=} \frac{1}{k} \sum_{m=1}^{\infty} \frac{C_m}{m!\binom{m+k-1}{m-1}} \;=\; \frac{1}{k} \sum_{m=0}^{\infty} \frac{C_{m+1}}{(m+1)!\binom{m+k}{m}}\,. && (2.1.7)
\end{aligned}
$$

Next, we prove a recurrence formula for the numbers C_m. To do this, we perform a folding of the Taylor series for $-\log(1-t)$ and $f(t)$. Again, let $t \in \mathbb{R}$ with $|t| < 1$. Then we have

$$-t \;=\; \frac{t}{\log(1-t)} \cdot \log(1-t) \;=\; \Big(\sum_{m=0}^{\infty} \frac{C_m}{m!} t^m \Big) \Big(\sum_{\mu=1}^{\infty} \frac{t^\mu}{\mu} \Big) \;=\; \sum_{\nu=1}^{\infty} \Big(\sum_{m=0}^{\nu-1} \frac{C_m}{m!(\nu-m)} \Big) t^\nu \,,$$

where $\nu = m + \mu$ is the new summation variable. Comparing coefficients on both sides of this identity, we obtain

$$C_0 \;=\; -1 \quad (\text{for } \nu = 1), \qquad \text{and} \qquad \sum_{m=0}^{\nu-1} \frac{C_m}{m!(\nu-m)} \;=\; 0 \quad (\text{for } \nu > 1).$$

Separating the term for $m = 0$ from the sum and then decreasing the summation variable by one, we have the recurrence formula

$$\frac{1}{\nu} \;=\; \sum_{m=0}^{\nu-2} \frac{C_{m+1}}{(m+1)!(\nu-m-1)} \qquad (\nu \geqslant 2)\,. \tag{2.1.8}$$

At this point, we will present the main result in this section. To do so, we need the following quantities: We define a sequence $\big(t_{m+2}(x)\big)_{m\geqslant 0}$ of polynomials and a sequence $\big(t_{m+2}\big)_{m\geqslant 0}$ of rational numbers:

$$t_2(x) \;:=\; \frac{x^2}{2}\,, \qquad t_{m+2}(x) \;:=\; \frac{1}{(m+1)!} \int_0^x u(1-u)(2-u)\cdots(m-u)\, du \quad (m \geqslant 1)\,; \tag{2.1.9}$$

$$t_2 \;:=\; \frac{1}{2}\,, \qquad t_{m+2} \;:=\; \frac{1}{(m+1)!} \int_0^1 u(1-u)(2-u)\cdots(m-u)\, du \quad (m \geqslant 1)\,. \tag{2.1.10}$$

In this way, $t_m = t_m(1)$ $(m \geqslant 2)$, and

$$t_3 \;=\; \frac{1}{12}\,, \quad t_4 \;=\; \frac{1}{24}\,, \quad t_5 \;=\; \frac{19}{720}\,, \quad t_6 \;=\; \frac{3}{160}\,, \quad t_7 \;=\; \frac{863}{60\,480}\,, \ldots$$

Lemma 1. *We have for every positive integer $k \geqslant 2$,*

$$\gamma \;=\; \frac{1}{k} \sum_{m=0}^{\infty} \frac{t_{m+2}}{\binom{m+k}{m}} + \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} \Big) - \log k\,. \tag{2.1.11}$$

The statement is also true for $k = 1$ if, in this case, the empty sum on the right-hand side of (2.1.11) is set to 0.

The lemma follows from (2.1.7) if we can show the identity

$$t_{m+2} \;=\; \frac{C_{m+1}}{(m+1)!} \tag{2.1.12}$$

for all integers $m \geqslant 0$. For the proof of (2.1.12) and for a few later arguments, we need two more lemmas. With a slightly different notation, we find formula (2.1.11) in [7, 0.131].

Lemma 2. (i) *We have for natural numbers w and μ with $k \geqslant 1$ and $0 \leqslant \mu \leqslant w$,*

$$\sum_{\nu=0}^{\mu} (-1)^\nu \binom{w}{\nu} \;=\; (-1)^\mu \binom{w-1}{\mu}\,. \tag{2.1.13}$$

In the case of $\mu = w$, the right-hand side of (2.1.13) vanishes.
(ii) *We have for natural numbers $v \geqslant 1$ and $w \geqslant 1$,*

$$\sum_{\mu=0}^{w} \frac{(-1)^\mu}{v+\mu}\binom{w}{\mu} \;=\; \frac{1}{(v+w)\binom{v+w-1}{w}} \;=\; \frac{w!}{v(v+1)\cdots(v+w)}\,. \tag{2.1.14}$$

(iii) *We have for natural numbers $v \geqslant 1$ and $w \geqslant 1$ with $0 \leqslant w < v$,*

$$\sum_{\mu=0}^{v-1} \frac{(-1)^\mu}{v-\mu}\binom{w}{\mu} \;=\; \frac{(-1)^w}{(v-w)\binom{v}{w}}\,. \tag{2.1.15}$$

Proof. The identity in (2.1.13) is given by [7, 0.15, no.4]. For the proofs of (2.1.14) and (2.1.15) we apply the Zeilberger algorithm to the sums, which we denote by z_w [9]: For (2.1.14) we obtain the recurrence formula

$$(w+v+1)z_{w+1} - (w+1)z_w \;=\; 0 \quad \text{with the initial value} \quad z_1 \;=\; \frac{1}{v} - \frac{1}{v+1} \;=\; \frac{1}{v(v+1)}$$

satisfied by the left and right-hand side of (2.1.14). Formula (2.1.14) is also a special case of the identity mentioned in Remark 8.5. of [11, p.68] in (8.11).
For (2.1.15) we have the recurrence formula

$$(w-v+1)z_{w+1} - (w+1)z_w \;=\; 0 \quad \text{with the initial value} \quad z_0 \;=\; \frac{1}{v}$$

satisfied by both sides of (2.1.15). □

Lemma 3. *We have for every integer $\nu \geqslant 2$,*

$$\frac{x(1-x)(2-x)\cdots(\nu-1-x)}{\nu!} \;=\; \frac{x}{\nu} - \sum_{m=0}^{\nu-2} \frac{t_{m+2}(x)}{\nu-m-1} \tag{2.1.16}$$

and

$$\frac{1}{\nu} \;=\; \sum_{m=0}^{\nu-2} \frac{t_{m+2}}{\nu-m-1}\,. \tag{2.1.17}$$

Proof. Applying the fundamental theorem of integral calculus, we have for integers $\nu \geqslant 2$:

$$\begin{aligned} t'_{\nu+1}(x) \;&=\; -\frac{(0-x)(1-x)\cdots(\nu-1-x)}{\nu!} \\ &=\; -\int_0^x \frac{1}{\nu!}\frac{d}{dt}\big[(0-t)(1-t)\cdots(\nu-1-t)\big]\,dt \\ &=\; \int_0^x \frac{1}{\nu!}\sum_{m=0}^{\nu-1} \frac{(0-t)(1-t)\cdots(\nu-1-t)}{m-t}\,dt\,. \end{aligned} \tag{2.1.18}$$

We now need the following polynomial identity:

$$\frac{1}{\nu!}\sum_{m=0}^{\nu-1} \frac{(0-t)(1-t)\cdots(\nu-1-t)}{m-t} \;=\; \frac{1}{\nu} + \sum_{m=0}^{\nu-2} \frac{(0-t)(1-t)\cdots(m-t)}{(\nu-m-1)(m+1)!}\,. \tag{2.1.19}$$

On both sides of (2.1.19) there are polynomials from $\mathbb{Q}[t]$ that have degree $\nu-1$. So the polynomial identity in (2.1.19) is shown if it holds for ν different values $t=w\in\{0,1,\ldots,\nu-1\}$. The left-hand side of (2.1.19) takes the following value for each such w:

$$\begin{aligned}
&\left[\frac{1}{\nu!}\sum_{m=0}^{\nu-1}\frac{(0-t)(1-t)\cdots(\nu-1-t)}{m-t}\right]_{t=w}\\
=\ &\frac{1}{\nu!}\big[(0-w)(1-w)\cdots(-1)\big]\cdots\big[1\cdot 2\cdots(\nu-1-w)\big]\\
=\ &\frac{(-1)^{w}w!(\nu-1-w)!}{\nu!}\ =\ \frac{(-1)^{w}}{(\nu-w)\binom{\nu}{w}}\,.
\end{aligned}\tag{2.1.20}$$

On the right-hand side of (2.1.19), we obtain

$$\begin{aligned}
&\left[\frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(0-t)(1-t)\cdots(m-t)}{(\nu-m-1)(m+1)!}\right]_{t=w}\\
=\ &\frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(0-w)(1-w)\cdots(m-w)}{(\nu-m-1)(m+1)!}\ =\ \frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(-1)^{m+1}w(w-1)\cdots(w-m)}{(\nu-m-1)(m+1)!}\\
=\ &\frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(-1)^{m+1}w!}{(\nu-m-1)(m+1)!(w-m-1)!}\ =\ \frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(-1)^{m+1}}{\nu-m-1}\binom{w}{m+1}\\
=\ &\frac{1}{\nu}+\sum_{m=1}^{\nu-1}\frac{(-1)^{m}}{\nu-m}\binom{w}{m}\ =\ \sum_{m=0}^{\nu-1}\frac{(-1)^{m}}{\nu-m}\binom{w}{m}\ =\ \frac{(-1)^{w}}{(\nu-w)\binom{\nu}{w}}\,.
\end{aligned}\tag{2.1.21}$$

The last identity follows from (2.1.15) in Lemma 2. Thus, (2.1.20) and (2.1.21) prove the equation in (2.1.19). Therefore, it follows from (2.1.18), (2.1.19) and (2.1.9),

$$\begin{aligned}
-\frac{(0-x)(1-x)\cdots(\nu-1-x)}{\nu!}\ &=\ \int_0^x\Big(\frac{1}{\nu}+\sum_{m=0}^{\nu-2}\frac{(0-t)(1-t)\cdots(m-t)}{(\nu-m-1)(m+1)!}\Big)\,dt\\
&=\ \frac{x}{\nu}+\sum_{m=0}^{\nu-2}\frac{1}{\nu-m-1}\int_0^x\frac{(0-t)(1-t)\cdots(m-t)}{(m+1)!}\,dt\\
&=\ \frac{x}{\nu}-\sum_{m=0}^{\nu-2}\frac{t_{m+2}(x)}{\nu-m-1}\,.
\end{aligned}$$

This proves the identity in (2.1.16) of Lemma 3. Due to $\nu\geqslant 2$, the left-hand side of (2.1.16) vanishes for $x=1$, and the equation obtained in this way from (2.1.16) can be rearranged to (2.1.17) due to $t_{m+2}(1)=t_{m+2}$. This completes the proof of Lemma 3. □

Setting $\nu=2$ in (2.1.8) and (2.1.17), we obtain

$$C_1\ =\ \frac{1}{2}\ =\ t_2\,.$$

Thus, by (2.1.8) and (2.1.17), the sequences $\big(C_{m+1}/(m+1)!\big)_{m\geqslant 0}$ and $(t_{m+2})_{m\geqslant 0}$ satisfy the same recurrence formula and they start with the same initial value $1/2$ for $m=0$. This proves (2.1.12) and shows that

$$C_m \;=\; \int_0^1 t(1-t)(2-t)\dots(m-1-t)\,dt \tag{2.1.22}$$

holds for all integers $m \geqslant 2$.

In order to estimate the convergence rate of the partial sums of the infinite series in (2.1.11), we introduce the quantity

$$R_{n,k} \;:=\; \frac{1}{k}\sum_{m=n+1}^{\infty} \frac{t_{m+2}}{\binom{m+k}{m}}\,. \tag{2.1.23}$$

for integers $k \geqslant 2$ and $n \geqslant 1$.

Lemma 4. *Let $n \geqslant 12$ and $k \geqslant 2$. Then we have*

$$\frac{1}{7nk(k+1)\binom{n+k}{k}} \;\leqslant\; R_{n,k} \;\leqslant\; \frac{1}{2k^2\binom{n+k}{k}}\,. \tag{2.1.24}$$

Proof. We begin by deriving an upper bound for $R_{n,k}$. First, starting with (2.1.10), we have

$$t_{m+2} \;\leqslant\; \frac{1}{(m+1)!}\int_0^1 u\big(1\cdot 2\cdot 3\cdots m\big)\,du \;=\; \frac{1}{2(m+1)}\,. \tag{2.1.25}$$

This is used to estimate $R_{n,k}$ in (2.1.23). We apply the identities from Lemma 2 several times.

$$\begin{aligned}
R_{n,k} &\leqslant \frac{1}{k}\sum_{m=n+1}^{\infty}\frac{1}{2(m+1)\binom{m+k}{m}}\\
&= \frac{k!}{2k}\sum_{m=n+1}^{\infty}\frac{1}{(m+1)^2(m+2)(m+3)\cdots(m+k)}\\
&\leqslant \frac{(k-1)!}{2}\sum_{m=n+1}^{\infty}\frac{1}{m(m+1)(m+2)(m+3)\cdots(m+k)}\\
&\overset{(2.1.14)}{=} \frac{(k-1)!}{2k!}\sum_{m=n+1}^{\infty}\sum_{\nu=0}^{k}\frac{(-1)^{\nu}}{m+\nu}\binom{k}{\nu} \;=\; \frac{1}{2k}\sum_{m=n}^{\infty}\sum_{\nu=0}^{k}\frac{(-1)^{\nu}}{m+1+\nu}\binom{k}{\nu} \qquad (2.1.26)\\
&= \frac{1}{2k}\sum_{\nu=0}^{k}\sum_{\mu=\nu}^{\infty}\frac{(-1)^{\nu}}{n+1+\mu}\binom{k}{\nu} \qquad (\mu := m-n+\nu)\\
&= \frac{1}{2k}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\min\{\mu,k\}}\frac{(-1)^{\nu}}{n+1+\mu}\binom{k}{\nu}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{1}{n+1+\mu}\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{k}{\nu} + \frac{1}{2k}\sum_{\mu=k}^{\infty}\frac{1}{n+1+\mu}\sum_{\nu=0}^{k}(-1)^{\nu}\binom{k}{\nu}\\
&\overset{(2.1.13)}{=} \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{(-1)^{\mu}}{n+1+\mu}\binom{k-1}{\mu} \overset{(2.1.14)}{=} \frac{1}{2k(n+k)\binom{n+k-1}{k-1}} \;=\; \frac{1}{2k^2\binom{n+k}{k}}\,. \qquad (2.1.27)
\end{aligned}$$

This proves the upper bound in Lemma 4. For the lower estimate, we apply the condition $m \geqslant n+1 \geqslant 13$ and

$$t_{m+2} \;\geqslant\; \frac{1}{(m+1)!}\int_0^1 u(1-u)(2-1)(3-1)\cdots(m-1)\,du \;=\; \frac{1}{6m(m+1)} \;\geqslant\; \frac{1}{7m(m-1)}\,. \tag{2.1.28}$$

For $n \geq 12$, it follows from (2.1.23) that

$$\begin{aligned} R_{n,k} \quad &\geqslant \quad \frac{1}{k}\sum_{m=n+1}^{\infty}\frac{1}{7m(m-1)\binom{m+k}{m}} \;=\; \frac{k!}{7k}\sum_{m=n+1}^{\infty}\frac{1}{(m-1)m(m+1)\cdots(m+k)} \\ &= \quad \frac{k!}{7k}\sum_{m=n}^{\infty}\frac{1}{m(m+1)(m+2)\cdots(m+k+1)} \\ &\overset{(2.1.14)}{=} \quad \frac{k!}{7k(k+1)!}\sum_{m=n}^{\infty}\sum_{\nu=0}^{k+1}\frac{(-1)^{\nu}}{m+\nu}\binom{k+1}{\nu} \\ &= \quad \frac{1}{7k(k+1)}\sum_{m=n-1}^{\infty}\sum_{\nu=0}^{k+1}\frac{(-1)^{\nu}}{m+1+\nu}\binom{k+1}{\nu}\,. \end{aligned} \tag{2.1.29}$$

Now, we proceed analogously for the proof of the upper bound of $R_{n,k}$ (starting from (2.1.26)), replacing n with $n-1$ and k with $k+1$ at the appropriate places. In this way, we finally obtain

$$\begin{aligned} R_{n,k} \quad &\geqslant \quad \frac{1}{7k(k+1)(n-1+k+1)\binom{n-1+k+1-1}{k+1-1}} \;=\; \frac{1}{7k(k+1)(n+k)\binom{n+k-1}{k}} \\ &= \quad \frac{1}{7nk(k+1)\binom{n+k}{k}} \qquad (n \geqslant 12,\ k \geqslant 2)\,. \end{aligned} \tag{2.1.30}$$

This completes the proof of Lemma 4. □

2.2. **Analytic and arithmetic results for some binomial coefficients.**

This section summarizes some results about binomial coefficients that will be used later. These include both, arithmetic properties and estimates. Subsequently, we denote the least common multiple of the numbers $1, 2, \ldots, n$ by $d(n)$. Furthermore, $\lfloor x \rfloor$ is the largest integer that is less than or equal to the real number x. The notation $a|b$ commonly used in number theory characterizes the integer a as a divisor of the integer b. Furthermore, we write $e(n,p)$ for the exponent of the prime number p in the canonical prime factorization of the natural number n. Our first goal is to prove the following lemma.

Lemma 5. *Let a, b, n_1, n_2 be natural numbers with $0 \leqslant n_2 \leqslant n_1 \leqslant b$. Then we have*

$$\binom{a+n_2}{n_2} \,\bigg|\, d(b)\binom{a+n_1}{n_1}\,. \tag{2.2.1}$$

For $n_1 = b$, we obtain the special case

$$\binom{a+n_2}{n_2} \,\bigg|\, d(b)\binom{a+b}{b} \qquad (0 \leqslant n_2 \leqslant b)\,. \tag{2.2.2}$$

Proof. First, we need an inequality in the context of the floor function $\lfloor x \rfloor$. Let α and β be real numbers. Then, the inequality

$$\lfloor \alpha - \beta \rfloor \geqslant \lfloor \alpha \rfloor - \lfloor \beta \rfloor - 1 \tag{2.2.3}$$

holds. If we set $\alpha = z_1 + \delta_1$ and $\beta = z_2 + \delta_2$ with $z_1, z_2 \in \mathbb{Z}$ and $0 \leqslant \delta_1, \delta_2 < 1$, it is easy to see that (2.2.3) is given by $\lfloor \alpha - \beta \rfloor = z_1 - z_2 = \lfloor \alpha \rfloor - \lfloor \beta \rfloor > \lfloor \alpha \rfloor - \lfloor \beta \rfloor - 1$ for $\delta_2 \leqslant \delta_1$. In the case of $\delta_1 < \delta_2$, we obtain $\lfloor \alpha - \beta \rfloor = z_1 - z_2 - 1 = \lfloor \alpha \rfloor - \lfloor \beta \rfloor - 1$.
Using (2.2.3), we now prove the following statement. Let p be a prime number. Then we have

$$e\Big(\binom{n_1}{n_2}, p \Big) \leqslant e\big(d(n_1), p\big) . \tag{2.2.4}$$

We will use the familiar formulas $e(n!, p) = \sum_{j \geqslant 1} \big[n/p^j \big]$ and $e\big(d(n), p\big) = \big[\log n / \log p \big]$ to obtain

$$\begin{aligned} e\Big(\binom{n_1}{n_2}, p \Big) &= e\big(n_1!, p\big) - e\big((n_1 - n_2)!, p\big) - e\big(n_2!, p\big) \\ = \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big\lfloor \frac{n_1}{p^j} \Big\rfloor &- \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big\lfloor \frac{n_1 - n_2}{p^j} \Big\rfloor - \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big\lfloor \frac{n_2}{p^j} \Big\rfloor \\ \overset{(2.2.3)}{\leqslant} \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big\lfloor \frac{n_1}{p^j} \Big\rfloor &- \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big(\Big\lfloor \frac{n_1}{p^j} \Big\rfloor - \Big\lfloor \frac{n_2}{p^j} \Big\rfloor - 1 \Big) - \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} \Big\lfloor \frac{n_2}{p^j} \Big\rfloor \\ = \sum_{1 \leqslant j \leqslant \frac{\log n_1}{\log p}} 1 &= \Big\lfloor \frac{\log n_1}{\log p} \Big\rfloor = e\big(d(n_1), p\big) , \end{aligned} \tag{2.2.5}$$

which proves (2.2.4). Choosing any prime number p and multiplying the identity

$$\frac{\binom{a + n_1}{n_1}}{\binom{a + n_2}{n_2}} = \frac{\binom{a + n_1}{n_1 - n_2}}{\binom{n_1}{n_2}} \tag{2.2.6}$$

with $d(b)$, we have the following estimate:

$$\begin{aligned} e\left(\frac{d(b)\binom{a + n_1}{n_1}}{\binom{a + n_2}{n_2}}, p \right) &\overset{(2.2.6)}{=} e\big(d(b), p\big) + e\Big(\binom{a + n_1}{n_1 - n_2}, p \Big) - e\Big(\binom{n_1}{n_2}, p \Big) \\ &\geqslant e\big(d(b), p\big) - e\Big(\binom{n_1}{n_2}, p \Big) \\ &\overset{(2.2.4)}{\geqslant} e\big(d(b), p\big) - e\big(d(n_1), p\big) \\ &= e\Big(\frac{d(b)}{d(n_1)}, p \Big) \geqslant 0 . \end{aligned} \tag{2.2.7}$$

The last inequality in (2.2.7) is guaranteed by the condition $n_1 \leqslant b$. Since the prime number p was chosen arbitrarily, Lemma 5 is proven by (2.2.7). ☐

Lemma 6. (i) *For every natural number* n *which is divisible by 32, the following inequality holds:*

$$\binom{77n/32}{n} \geqslant \frac{c_1^n}{2\sqrt{n}} \quad \textit{with} \quad c_1 \,:=\, 5.121\,. \tag{2.2.8}$$

(ii) *If* $n \geq 448$ *is a natural number which is divisible by 32, the following inequalities hold:*

$$\frac{175}{2544}(45n+32)\binom{77n/32}{n} \;<\; \binom{5n/2}{n}, \tag{2.2.9}$$

$$7(45n+32)\binom{77n/32}{n} \;<\; \binom{23n/8}{n}. \tag{2.2.10}$$

(iii) *For every natural number* $n \geqslant 1$*, we have*

$$\sqrt{\frac{1}{6\pi n}}\Big(\frac{256}{27}\Big)^n \;<\; \binom{4n}{n} \;<\; \Big(\frac{256}{27}\Big)^n \tag{2.2.11}$$

Proof. We estimate the binomial coefficient on the left in (2.2.1) from below. Here we use a version of Stirling's formula due to H.Robbins [12]: We have for all integers $m \geqslant 1$,

$$\begin{aligned}\sqrt{2\pi m}\Big(\frac{m}{e}\Big)^m \;&<\; \sqrt{2\pi m}\Big(\frac{m}{e}\Big)^m e^{1/(12m+1)} \;<\; m! \\ &<\; \sqrt{2\pi m}\Big(\frac{m}{e}\Big)^m e^{1/(12m)} < \sqrt{2\pi(m+1)}\Big(\frac{m}{e}\Big)^m, \end{aligned} \tag{2.2.12}$$

where the right-hand inequality follows from $\sqrt{m}\,e^{1/(12m)} \;<\; \sqrt{m}\big(1+1/(6m)\big) \;<\; \sqrt{m+1}$. This gives

$$\begin{aligned}\binom{77n/32}{n} \;&=\; \frac{(77n/32)!}{n!(45n/32)!} \;>\; \frac{\sqrt{2\pi(77n/32)}}{\sqrt{2\pi(n+1)}\sqrt{2\pi(45n/32+1)}}\left(\frac{(77/32)^{77/32}}{(45/32)^{45/32}}\right)^n \\ &=\; \sqrt{\frac{77n}{2\pi(n+1)(45n+32)}}\left(\frac{(77/32)^{77/32}}{(45/32)^{45/32}}\right)^n. \end{aligned} \tag{2.2.13}$$

Since

$$(n+1)(45n+32) \;<\; 48n^2 \quad \text{holds if and only if} \quad n \;>\; \frac{77+\sqrt{6313}}{6} \;=\; 26.075\ldots\,,$$

we obtain for $n \geqslant 32$ from (2.2.13),

$$\binom{77n/32}{n} \;>\; \frac{\sqrt{77/(96\pi)}}{\sqrt{n}}\left(\frac{(77/32)^{77/32}}{(45/32)^{45/32}}\right)^n \;>\; \frac{(5.121)^n}{2\sqrt{n}}\,. \tag{2.2.14}$$

This proves the inequality in (2.2.8).

If $r > 1$ is a rational number with $rn \in \mathbb{N}$, then, following the lines of proof of (2.2.8), we obtain from (2.2.12) the inequalities

$$\sqrt{\frac{rn}{2\pi(n+1)((r-1)n+1)}}\cdot\Big(\frac{r^r}{(r-1)^{r-1}}\Big)^n \;<\; \binom{rn}{n} \;<\; \sqrt{\frac{rn+1}{2\pi n(r-1)n}}\cdot\Big(\frac{r^r}{(r-1)^{r-1}}\Big)^n. \tag{2.2.15}$$

We simplify the terms under the root signs using the inequalities $n+1 \leqslant 2n$ and $rn+1 \leqslant 2rn$. We assume the condition

$$n \geqslant \frac{1}{r-1}\,, \tag{2.2.16}$$

in order to estimate the term on the left-hand side in (2.2.15). This implies the inequality $(r-1)n+1 \leqslant 2(r-1)n$ and is satisfied for all $r>1$. Then, we obtain from (2.2.15),

$$\sqrt{\frac{r}{8\pi(r-1)n}} \cdot \Big(\frac{r^r}{(r-1)^{r-1}} \Big)^n \;<\; \binom{rn}{n} \;<\; \sqrt{\frac{r}{\pi(r-1)n}} \cdot \Big(\frac{r^r}{(r-1)^{r-1}} \Big)^n . \tag{2.2.17}$$

To prove (2.2.9), we apply (2.2.17) twice. First with $r=77/32$, then with $r=5/2$:

$$\begin{aligned}
\frac{175}{2544}(45n+32)\binom{77n/32}{n} \;&\overset{(r=77/32)}{<}\; \frac{175}{2544} \cdot 46n \cdot \sqrt{\frac{77/32}{\pi \cdot 45n/32}} \cdot \left(\frac{(77/32)^{77/32}}{(45/32)^{45/32}} \right)^n \\
&<\; \frac{4025}{1272}\sqrt{\frac{77}{45\pi}} \cdot \sqrt{n} \cdot \Big(\frac{26}{5} \Big)^n \\
&\overset{(n\geqslant 448)}{<}\; \sqrt{\frac{5}{24\pi n}} \cdot \Big(\frac{53}{10} \Big)^n \\
&<\; \sqrt{\frac{5/2}{8\pi \cdot 3n/2}} \cdot \left(\frac{(5/2)^{5/2}}{(3/2)^{3/2}} \right)^n \\
&\overset{(r=5/2)}{<}\; \binom{5n/2}{n}\,;
\end{aligned} \tag{2.2.18}$$

Note that (2.2.16) is obviously fullfilled here for $r=5/2$. The inequality in (2.2.10) is proven similarly:

$$\begin{aligned}
7(45n+32)\binom{77n/32}{n} \;&\overset{(r=77/32)}{<}\; 7 \cdot 46 \cdot \sqrt{\frac{77}{45\pi}} \cdot \sqrt{n} \cdot \Big(\frac{26}{5} \Big)^n \;\overset{(n\geqslant 448)}{<}\; \sqrt{\frac{23}{120\pi n}} \cdot \Big(\frac{32}{5} \Big)^n \\
&<\; \sqrt{\frac{23/8}{8\pi \cdot 15n/8}} \cdot \Big(\frac{(23/8)^{23/8}}{(15/8)^{15/8}} \Big)^n \\
&\overset{(r=23/8)}{<}\; \binom{23n/8}{n} .
\end{aligned} \tag{2.2.19}$$

For the proof of (2.2.11), we use both sides of (2.2.17) with $r=4$:

$$\sqrt{\frac{1}{6\pi n}}\Big(\frac{256}{27} \Big)^n \;<\; \binom{4n}{n} \;<\; \sqrt{\frac{4}{3\pi n}}\Big(\frac{256}{27} \Big)^n \;<\; \Big(\frac{256}{27} \Big)^n .$$

Lemma 6 is now proven completely . ☐

We conclude this section with the statement of a quantitative estimate for $d(m)$.

Lemma 7. *For every natural number* $m \geqslant 5 \cdot 10^{343}$, *we have*

$$2.71^m \;<\; d(m) \;<\; 2.72^m . \tag{2.2.20}$$

Proof. It is well-known that the number $d(n)$ can be expressed by

$$d(m) \;=\; e^{\theta(m)}\,,$$

where

$$\theta(m) \;:=\; \sum_{\substack{p\,\leqslant\, m\\ p\,\in\,\mathbb{P}}} \Big\lfloor \frac{\log m}{\log p} \Big\rfloor \log p$$

with the set $\mathbb{P}$ of all prime numbers. By the prime number theorem we have $\theta(m) \sim m$. J.B.Rosser and L.Schoenfeld [13] have published explicit bounds for $\theta(m)$:

$$m\Big(1-\frac{1}{2\log m}\Big) \;<\; \theta(m) < m\Big(1+\frac{1}{2\log m}\Big) \qquad (m \geqslant 563)\,.$$

This implies for $m \geqslant 5\cdot 10^{343}$,

$$\log d(m) \;<\; m\Big(1+\frac{1}{2\log m}\Big) \;<\; m\Big(1+\frac{1}{2\log(5\cdot 10^{343})}\Big) \;<\; m\log\big(2.72\big)$$

and

$$\log d(m) \;>\; m\Big(1-\frac{1}{2\log m}\Big) \;>\; m\Big(1-\frac{1}{2\log(5\cdot 10^{343})}\Big) \;>\; m\log\big(2.71\big)\,.$$

The lemma is proven. □

2.3. **Small solutions of systems of vanishing and non-vanishing linear forms: the first fundamental lemma.**

Let M and N be natural numbers with $N > M$. We consider the linear forms

$$L_\mu(x_0,\dots,x_N) \;\;:=\;\; \sum_{\nu=0}^{N} a_{\mu\nu}x_\nu \qquad (\mu = 1,\dots,M)\,, \tag{2.3.1}$$

$$L_{M+1}(x_0,\dots,x_N) \;\;:=\;\; \sum_{\nu=0}^{N} b_\nu x_\nu \tag{2.3.2}$$

in the variables $x_0,\dots,x_N$ and with integer coefficients $a_{\mu\nu}$ and b_ν. Furthermore, we define the quantities $A_1,\dots,A_M$ by

$$A_\mu \;:=\; \max\big\{1,\,|a_{\mu\nu}|\;:\;0\leqslant \nu\leqslant N\big\} \qquad (1\leqslant \mu\leqslant M)\,. \tag{2.3.3}$$

Let $n > 1$ be a natural number and $c_2, c_3 > 1$ be certain real numbers. Then, we assume the following conditions on the linear forms $L_1, \ldots, L_{M+1}$:

$$\frac{25n}{16} \; < \; N \; < \; \frac{51n}{32} \; < \; 2n\,, \tag{2.3.4}$$

$$M \; < \; \frac{(1+\varepsilon)n}{\log n} \quad \text{with} \quad \varepsilon \; := \; 1/50\,000\,, \tag{2.3.5}$$

$$A_1 \; < \; c_2^n n!\,, \tag{2.3.6}$$

$$A_\mu \; < \; c_3 \log n \qquad (2 \leqslant \mu \leqslant M)\,, \tag{2.3.7}$$

$$b_\nu \; \ll \; 1 \qquad (0 \leqslant \nu \leqslant N)\,. \tag{2.3.8}$$

The constant in the $\ll$ symbol is absolute. (2.3.6) and (2.3.7) show that the first linear form L_1 has significantly larger coefficients than all other linear forms in (2.3.1).

For the formulation of the following fundamental lemma, we still need the notation $|\mathcal{M}|$ for a matrix $\mathcal{M}$, which denotes the greatest absolute value of the elements of the matrix $\mathcal{M}$.

Lemma 8. *(First fundamental lemma)*
Let the linear form $L_{M+1}(x_0, \ldots, x_N)$ of the system (2.3.1) and (2.3.2) be linearly independent over $\mathbb{Q}$ from the M linear forms $L_\mu(x_0, \ldots, x_N)$ in (2.3.1). Let (2.3.3) to (2.3.8) be assumed. Furthermore, let $\mathcal{C} \in \mathbb{Z}^{(N+1)\times(N+1)}$ be a regular matrix, and let $\vec{a}_\mu \in \mathbb{Z}^{1\times(N+1)}$ $(1 \leqslant \mu \leqslant M)$ denote the row vector of coefficients of the μ-th equation of the system (2.3.1). Furthermore, let $D_\mu := |\vec{a}_\mu \mathcal{C}|$, and let ω_μ $(2 \leqslant \mu \leqslant M)$ denote the number of non-vanishing coefficients of the μ-th equation of the system (2.3.1). With a positive constant c_4, we assume that $\omega_\mu \leqslant c_4 N/\mu$ $(2 \leqslant \mu \leqslant M)$ holds. Then the linear system of M equations and one inequality,

$$\begin{aligned} L_\mu(x_0, \ldots, x_N) \; &= \; 0 \qquad (1 \leqslant \mu \leqslant M)\,, && (2.3.9)\\ L_{M+1}(x_0, \ldots, x_N) \; &\neq \; 0\,, && (2.3.10)\end{aligned}$$

has an integer solution $(x_0, \ldots, x_N) \in \mathbb{Z}^{N+1}$ with

$$\bigl|x_\nu\bigr| \; \leqslant \; 2n|\mathcal{C}|^{1+M/2}\bigl(c_5 e\sqrt{2}\bigr)^{M/2}\bigl(\log n\bigr)^{M/2} n^{M/2}\sqrt{D_1}\bigl(\max\bigl\{D_1,\, c_5 n \log(n)\,|\mathcal{C}|\,\bigr\}\bigr)^{M/2} \tag{2.3.11}$$

for all $\nu = 0, \ldots, N$, where $c_5 := 2c_3c_4$.

To prove Lemma 8, we need further auxiliary theorems.

Lemma 9. *Let*

$$\mathcal{A} \; := \; \begin{pmatrix} a_{1,0} & \ldots & a_{1,N} \\ \vdots & & \vdots \\ a_{M,0} & \ldots & a_{M,N} \\ b_0 & \ldots & b_N \end{pmatrix} \; \in \; \mathbb{Z}^{(M+1)\times(N+1)} \tag{2.3.12}$$

be the coefficient matrix of the system (2.3.1), (2.3.2). Let the last row of $\mathcal{A}$ be linearly independent over $\mathbb{Q}$ from the first M rows of $\mathcal{A}$. Furthermore, let the matrix $\mathcal{C}$ be given from

Lemma 8 by

$$\mathcal{C} := \begin{pmatrix} c_{0,1} & \cdots & c_{0,N+1} \\ \vdots & & \vdots \\ c_{N,1} & \cdots & c_{N,N+1} \end{pmatrix} \in \mathbb{Z}^{(N+1)\times(N+1)} \tag{2.3.13}$$

as a square matrix with full rank. Then the last row of $\mathcal{AC} \in \mathbb{Z}^{(M+1)\times(N+1)}$ *is linearly independent over* $\mathbb{Q}$ *from the first* M *rows of the matrix* $\mathcal{AC}$*.*

The somewhat unusual indexing of the elements of $\mathcal{C}$ is due to the later application.

Proof. First, we introduce a few more notations. Let

$$\mathcal{D} := \mathcal{AC} = \begin{pmatrix} d_{1,0} & \cdots & d_{1,N} \\ \vdots & & \vdots \\ d_{M,0} & \cdots & d_{M,N} \\ e_0 & \cdots & e_N \end{pmatrix} \in \mathbb{Z}^{(M+1)\times(N+1)}\,. \tag{2.3.14}$$

We also need the following vectors:

$$\begin{array}{lcll} \vec{a}_\mu & := & \big(a_{\mu,0},\dots,a_{\mu,N}\big) & (1 \leqslant \mu \leqslant M)\,, \\ \vec{b} & := & \big(b_0,\dots,b_N\big)\,, & \\ \vec{c}_\nu & := & \big(c_{0,\nu},\dots,c_{N,\nu}\big)^T & (1 \leqslant \nu \leqslant N+1)\,, \\ \vec{d}_\mu & := & \vec{a}_\mu \mathcal{C} := \big(d_{\mu,0},\dots,d_{\mu,N}\big) & (1 \leqslant \mu \leqslant M)\,, \\ \vec{e} & := & \big(e_0,\dots,e_N\big)\,. & \end{array}$$

Let us assume that the vector $\vec{e}$ is linearly dependent over $\mathbb{Q}$ from the vectors $\vec{d}_1,\dots,\vec{d}_M$. Consequently, rational numbers $\lambda_1,\dots,\lambda_M$ exist in such a way that

$$\vec{e} = \sum_{\mu=1}^{M} \lambda_\mu \vec{d}_\mu\,. \tag{2.3.15}$$

In the following, we use the notation $\langle\, \vec{x}, \vec{y}\,\rangle$ for the scalar product of two vectors $\vec{x}$ and $\vec{y}$ of the same dimension. Due to (2.3.12), (2.3.13) and (2.3.14), we then have

$$d_{\mu,\nu} = \langle\, \vec{a}_\mu, \vec{c}_{\nu+1}\,\rangle\,, \tag{2.3.16}$$

$$e_\nu = \langle\, \vec{b}, \vec{c}_{\nu+1}\,\rangle \tag{2.3.17}$$

for $1 \leqslant \mu \leqslant M$ and $0 \leqslant \nu \leqslant N$. Thus, (2.3.15) can be rewritten as

$$\vec{e} = \sum_{\mu=1}^{M} \lambda_\mu \big(d_{\mu,0},\dots,d_{\mu,N}\big) = \sum_{\mu=1}^{M} \lambda_\mu \big(\langle\, \vec{a}_\mu, \vec{c}_1\,\rangle,\dots,\langle\, \vec{a}_\mu, \vec{c}_{N+1}\,\rangle\big). \tag{2.3.18}$$

Applying (2.3.15) to (2.3.17), we obtain

$$\begin{aligned}
\vec{0} &= \big(\langle\,\vec{b},\vec{c}_1\,\rangle,\dots,\langle\,\vec{b},\vec{c}_{N+1}\,\rangle\big) - \sum_{\mu=1}^{M}\big(\langle\,\lambda_\mu\vec{a}_\mu,\vec{c}_1\,\rangle,\dots,\langle\,\lambda_\mu\vec{a}_\mu,\vec{c}_{N+1}\,\rangle\big)\\
&= \left(\langle\,\vec{b},\vec{c}_1\,\rangle - \sum_{\mu=1}^{M}\langle\,\lambda_\mu\vec{a}_\mu,\vec{c}_1\,\rangle,\dots,\ \langle\,\vec{b},\vec{c}_{N+1}\,\rangle - \sum_{\mu=1}^{M}\langle\,\lambda_\mu\vec{a}_\mu,\vec{c}_{N+1}\,\rangle\right)\\
&= \left(\langle\,\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu,\vec{c}_1\,\rangle,\dots,\langle\,\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu,\vec{c}_{N+1}\,\rangle\right)\\
&= \big(\,\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu\,\big)\cdot\big(\,\vec{c}_1,\dots,\vec{c}_{N+1}\,\big)\\
&= \big(\,\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu\,\big)\cdot\mathcal{C}\,. \qquad (2.3.19)
\end{aligned}$$

This is equivalent with

$$\mathcal{C}^T\cdot\left(\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu\right)^T = \vec{0}^{\,T}\,. \qquad (2.3.20)$$

We interpret this matrix - vector equation as a quadratic, homogeneous system of equations with the coefficient matrix $\mathcal{C}^T$ solved by

$$\left(\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu\right)^T\,.$$

Since the quadratic matrix $\mathcal{C}^T$ has full rank, it follows that

$$\vec{b}-\sum_{\mu=1}^{M}\lambda_\mu\vec{a}_\mu = \vec{0}\,.$$

This contradicts the assumed linear independence over $\mathbb{Q}$ of the vector $\vec{b}$ from the vectors $\vec{a}_1,\dots,\vec{a}_M$. The contradiction proves the lemma. □

We now need a variant of Siegel's lemma. Let M and N be natural numbers with $N>M$. We are now looking for small non-trivial integer solutions of the homogeneous linear system of equations

$$\sum_{\nu=0}^{N} a_{\mu\nu}x_\nu = 0 \qquad (1\leqslant\mu\leqslant M)\,, \qquad (2.3.21)$$

where all coefficients $a_{\mu\nu}$ are integers. Setting

$$D := \max\big\{|a_{\mu\nu}|\,:\,0\leqslant\nu\leqslant N,\,1\leqslant\mu\leqslant M\big\}\,,$$

the classical lemma of Siegel guarantees the existence of a non-trivial integer solution of the system (2.3.21) that satisfies the inequalities

$$|x_\nu| \leqslant \big(D(N+1)\big)^{M/(N-M+1)} \qquad (0\leqslant\nu\leqslant N)\,, \qquad (2.3.22)$$

cf. [5, Lemma 1, ch. 2.4]. If the size of the coefficients $a_{\mu\nu}$ varies extremely between the individual equations of the system (2.3.21), then a smaller upper bound can be expected for the components x_ν of a non-trivial integer solution. For our purposes, the bound in (2.3.22) is too large; we need a sharper result due to E. Bombieri and J. Vaaler [3].

Lemma 10. *For a linear system of equations with M' linearly independent equations and $N' > M'$ unknowns $t_1, \dots, t_{N'}$, there exists a non-trivial integer solution $t_1, \dots, t_{N'}$ whose components satisfy the inequalities*

$$|t_\nu| \leqslant \left(G^{-1} \sqrt{\left| \det(\mathcal{H}\mathcal{H}^T) \right|} \right)^{1/(N'-M')} \qquad (1 \leqslant \nu \leqslant N')\,.$$

Here, $\mathcal{H}$ denotes the coefficient matrix of the system of equations, and G is the greatest common divisor of all determinants of the $M' \times M'$ minors of $\mathcal{H}$.

Now, we consider a homogeneous system of equations that is extended by a linearly independent equation to form a system of equations and one inequality:

$$\left.\begin{array}{rcll} \sum\limits_{\nu=0}^{N} d_{\mu\nu} x_\nu & = & 0 & \quad (1 \leqslant \mu \leqslant M)\,, \\ \sum\limits_{\nu=0}^{N} e_\nu x_\nu & \neq & 0\,, & \end{array}\right\} \tag{2.3.23}$$

where $e_0, \dots, e_N$ are also integers. Again, we assume that $N > M$.

Lemma 11. *Let the linear form $L'_{M+1} := e_0 x_0 + \cdots + e_N x_N$ in (2.3.23) be linearly independent over $\mathbb{Q}$ from the M linear forms $L'_\mu := d_{\mu,0} x_0 + \cdots + d_{\mu,N} x_N$ $(1 \leqslant \mu \leqslant M)$. Furthermore, let $D_1, \dots, D_M$ be defined by*

$$D_\mu := \max_{0 \leq \nu' \leq N} \left| d_{\mu\nu'} \right| \qquad (1 \leqslant \mu \leqslant M)\,.$$

Let all the conditions for the system (2.3.1) and (2.3.2) be satisfied as in Lemma 8. Here, the system in (2.3.23) is obtained from the system in (2.3.1) and (2.3.2) by the transformation $\mathcal{A}\mathcal{C}$, where $\mathcal{A}$ and $\mathcal{C}$ are given by (2.3.12) and (2.3.13). Then there exists an integer solution $x_0, \dots, x_N$ of the system (2.3.23), whose components satisfy the inequalities

$$|x_\nu| \leq \left(c_5 e \sqrt{2}\right)^{M/2} \left(\log n\right)^{M/2} n^{M/2} |\mathcal{C}|^{M/2} \sqrt{D_1} \left(\max\left\{ D_1,\, c_5 n \log(n)\, |\mathcal{C}| \right\} \right)^{M/2} \quad (0 \leqslant \nu \leqslant N)\,.$$

Proof. At the beginning of our proof, we apply a concept due to Paul Georg Becker-Landeck from his proof of Lemma 4 in [2] and combine it with the result of E. Bombieri and J. Vaaler from Lemma 10.

Let us consider the coefficient matrix

$$\mathcal{D} := \begin{pmatrix} d_{1,0} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{M,0} & \dots & d_{M,N} \\ e_0 & \dots & e_N \end{pmatrix}. \tag{2.3.24}$$

of all linear forms $L'_1, \dots, L'_{M+1}$. Furthermore, let $m' := \operatorname{rg} \mathcal{D} \leqslant M+1$. Due to the linear independence of L'_{M+1} from the forms $L'_1, \dots, L'_M$, the matrix

$$\mathcal{D}_1 := \begin{pmatrix} d_{1,0} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{M,0} & \dots & d_{M,N} \end{pmatrix} . \tag{2.3.25}$$

has rank $m'-1$. In $\mathcal{D}_1$, we first remove all rows that are linearly dependent from $m'-1$ linearly independent rows. Without limiting the generality, we are removing the rows $m', m'+1, \dots, M$ from $\mathcal{D}_1$. In this way, we obtain the matrix

$$\mathcal{D}_2 := \begin{pmatrix} d_{1,0} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{m'-1,0} & \dots & d_{m'-1,N} \end{pmatrix} . \tag{2.3.26}$$

The matrix

$$\mathcal{D}_3 := \begin{pmatrix} d_{1,0} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{m'-1,0} & \dots & d_{m'-1,N} \\ e_0 & \dots & e_N \end{pmatrix} \tag{2.3.27}$$

has rank m', since the last row is still linearly independent from the remaining rows. Since the column rank of $\mathcal{D}_3$ is also m', we can remove such columns in $\mathcal{D}_3$ due to $N \geq M+1 \geq m'$ (i.e., a priori the corresponding unknowns x_ν in (2.3.23) are set to zero) that are linearly dependent from m' remaining columns, so that the row and column ranks are again m'. Without restricting the generality, we remove the columns $m', m'+1, \dots, N$ from $\mathcal{D}_3$. In this way, we obtain the matrix

$$\mathcal{D}_4 := \begin{pmatrix} d_{1,0} & \dots & d_{1,m'-1} \\ \vdots & & \vdots \\ d_{m'-1,0} & \dots & d_{m'-1,m'-1} \\ e_0 & \dots & e_{m'-1} \end{pmatrix} . \tag{2.3.28}$$

The $m' \times m'$ quadratic matrix $\mathcal{D}_4$ has full rank. We apply Lemma 10 on the linear system of equations

$$\sum_{\nu=0}^{m'-1} d_{\mu\nu} x_\nu = 0 \qquad (1 \leq \mu \leq m'-1) \tag{2.3.29}$$

with its matrix of coefficients,

$$\mathcal{D}_5 := \begin{pmatrix} d_{1,0} & \dots & d_{1,m'-1} \\ \vdots & & \vdots \\ d_{m'-1,0} & \dots & d_{m'-1,m'-1} \end{pmatrix} . \tag{2.3.30}$$

In doing so, we identify the parameters using $M' \to m'-1$, $N' \to m'$ and $\mathcal{H} \to \mathcal{D}_5$. Because $N' - M' = m' - (m'-1) = 1$, we obtain a non-trivial integer solution whose components satisfy the inequality

$$|x_\nu| \leqslant \sqrt{\left| \det(\mathcal{D}_5 \mathcal{D}_5^T) \right|} \qquad (0 \leqslant \nu \leqslant m'-1) . \tag{2.3.31}$$

The components $x_{m'} = \cdots = x_N = 0$, which were set to zero, trivially satisfy this inequality, so that we have

$$|x_\nu| \leqslant \sqrt{\left|\det(\mathcal{D}_5\mathcal{D}_5^T)\right|} \qquad (0 \leqslant \nu \leqslant N)\,. \tag{2.3.32}$$

This solution $\vec{x} := (x_0, \ldots, x_N)^T \neq \vec{0}$ also satisfies the lower inequality in (2.3.23), because due to the full rank of the quadratic matrix $\mathcal{D}_4$, the inequality

$$\mathcal{D}_4\vec{x} \neq \vec{0} \tag{2.3.33}$$

holds for any non-zero vector $\vec{x} \in \mathbb{Z}^{m'}$. Then, making use of (2.3.29), we obtain the desired inequality

$$e_0x_0 + \cdots + e_Nx_N \;=\; e_0x_0 + \cdots + e_{m'}x_{m'} \;\neq\; 0\,. \tag{2.3.34}$$

Finally, we enlarge the right-hand side of the inequality in (2.3.32). To do this, we denote the elements of the $(m'-1) \times (m'-1)$ matrix $\mathcal{D}_5\mathcal{D}_5^T$ by

$$\mathcal{D}_5\mathcal{D}_5^T \;=:\; (h_{\mu\nu})_{1\leqslant\mu,\nu\leqslant m'-1}\,. \tag{2.3.35}$$

Due to the matrix multiplication and to the construction of $\mathcal{D}_5$, we have the inequalities

$$\left|h_{\mu\nu}\right| \leqslant (m'-1)D_\mu D_\nu \qquad (1 \leqslant \mu, \nu \leqslant m'-1)\,, \tag{2.3.36}$$

where D_μ was introduced in Lemma 8 for $\mu = 1, \ldots, M$. We temporarily introduce the quantity $k := m' - 1$. For the following estimates, we need some preliminary considerations: According to (2.3.7), it is $A_\mu < c_3 \log n$ for $2 \leqslant \mu \leqslant M$. Using the quantities introduced in and before Lemma 8, we have

$$D_\mu \;=\; |\vec{a}_\mu\mathcal{C}| \;\leqslant\; \omega_\mu A_\mu\,|\mathcal{C}| \;\leqslant\; \frac{c_3c_4N\log(n)\,|\mathcal{C}|}{\mu} \;\leqslant\; \frac{2c_3c_4n\log(n)\,|\mathcal{C}|}{\mu} \qquad (2 \leqslant \mu \leqslant M)\,.$$

Hadamard's inequality together with (2.3.36), $k = m' - 1 \leqslant M$ and application of the Euclidean vector norm $|.|_2$ allows to estimate $\big|\det(\mathcal{D}_5\mathcal{D}_5^T)\big|$ in the following way:

$$\begin{aligned}
\big|\det(\mathcal{D}_5\mathcal{D}_5^T)\big| &\leqslant \prod_{\nu=1}^{k} \Big|(h_{1,\nu},\dots,h_{k,\nu})^T\Big|_2 = \prod_{\nu=1}^{k}\sqrt{\sum_{\mu=1}^{k} h_{\mu\nu}^2} \overset{(2.3.36)}{\leqslant} \prod_{\nu=1}^{k}\sqrt{\sum_{\mu=1}^{k} k^2 D_\mu^2 D_\nu^2} \\
&= k^k \prod_{\nu=1}^{k}\sqrt{\sum_{\mu=1}^{k} D_\mu^2 D_\nu^2} = k^k D_1 \sqrt{\sum_{\mu=1}^{k} D_\mu^2} \cdot \prod_{\nu=2}^{k}\sqrt{\sum_{\mu=1}^{k} D_\mu^2 D_\nu^2} \\
&= k^k D_1 \sqrt{\Big(D_1^2 + \sum_{\mu=2}^{k} D_\mu^2\Big)} \cdot \prod_{\nu=2}^{k}\sqrt{D_\nu^2\Big(D_1^2 + \sum_{\mu=2}^{k} D_\mu^2\Big)} \\
&\leqslant k^k D_1 \sqrt{D_1^2 + \sum_{\mu=2}^{k} \frac{4c_3^2c_4^2n^2\log^2(n)|\mathcal{C}|^2}{\mu^2}} \\
&\quad \times \prod_{\nu=2}^{k}\sqrt{\frac{4c_3^2c_4^2n^2\log^2(n)|\mathcal{C}|^2}{\nu^2}\Big(D_1^2 + \sum_{\mu=2}^{k}\frac{4c_3^2c_4^2n^2\log^2(n)|\mathcal{C}|^2}{\mu^2}\Big)} \\
&< k^k D_1 \sqrt{\big(D_1^2 + c_5^2n^2\log^2(n)|\mathcal{C}|^2\big)\cdot\frac{\big(c_5^2n^2\log^2(n)|\mathcal{C}|^2\big)^{k-1}}{(k!)^2}\cdot\prod_{\nu=2}^{k}\big(D_1^2 + c_5^2n^2\log^2(n)|\mathcal{C}|^2\big)} \\
&= k^k D_1 \frac{\big(c_5 n\log(n)\,|\mathcal{C}|\big)^{k-1}}{k!}\sqrt{\big(D_1^2 + c_5^2n^2\log^2(n)|\mathcal{C}|^2\big)^k} \\
&\leqslant \frac{k^k}{k!} D_1 \big(c_5 n\log(n)\,|\mathcal{C}|\big)^{k-1}\big(2\max\big\{D_1^2,\, c_5^2n^2\log^2(n)|\mathcal{C}|^2\big\}\big)^{k/2}. \qquad (2.3.37)
\end{aligned}$$

Here, we have set $c_5 := 2c_3c_4$; note that $\sum_{\mu=2}^{\infty}\mu^{-2} = \pi^2/6 - 1 < 1$. Using Stirling's inequality (2.2.12), we further obtain:

$$\begin{aligned}
\big|\det(\mathcal{D}_5\mathcal{D}_5^T)\big| &\leqslant \frac{D_1}{\sqrt{2\pi k}}\big(c_5 e\sqrt{2}\big)^k\big(\log n\big)^k n^k|\mathcal{C}|^k\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^k \\
&\leqslant \big(c_5 e\sqrt{2}\big)^M\big(\log n\big)^M n^M|\mathcal{C}|^M D_1\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^M \qquad (2.3.38)
\end{aligned}$$

because $k \leqslant m' - 1 \leqslant M$. If we substitute (2.3.38) into (2.3.32), we obtain for $x_0,\dots,x_N$,

$$|x_\nu| \leqslant \big(c_5 e\sqrt{2}\big)^{M/2}\big(\log n\big)^{M/2} n^{M/2}|\mathcal{C}|^{M/2}\sqrt{D_1}\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^{M/2}. \qquad (2.3.39)$$

Overall, $x_{m'} = x_{m'+1} = \dots = x_N = 0$ as well as (2.3.29), (2.3.34) and (2.3.39) prove Lemma11 for an integer solution $x_0,\dots,x_N$ of the system in (2.3.23). □

Proof of Lemma 8. The matrix $\mathcal{A}$ in (2.3.12) represents the coefficient matrix of the system from the linear forms in (2.3.1) and (2.3.2). Since, according to a condition in Lemma 8, the last row of $\mathcal{A}$ is linearly independent from the other rows. Since the matrix $\mathcal{C}$ is regular, it follows from Lemma 9, that the last row of the matrix $\mathcal{D} = \mathcal{A}\mathcal{C}$ is also linearly independent from the other rows. Let the elements of the matrix $\mathcal{D}$ be denoted as in (2.3.14). We write D_μ for the maximum norm of the μ-th row vector of $\mathcal{D}$ ($1 \leqslant \mu \leqslant M$) as in Lemma 11, which can be applied to the system given by (2.3.23): The system (2.3.23) has an integer solution $\vec{x}\,' = \big(x'_0, \ldots, x'_N\big)^T$, whose components x'_ν satisfy the inequalities

$$|x'_\nu| \,\le\, \big(c_5 e\sqrt{2}\big)^{M/2}\big(\log n\big)^{M/2} n^{M/2}|\mathcal{C}|^{M/2}\sqrt{D_1}\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^{M/2} \;\; (0 \leqslant \nu \leqslant N)\,. \tag{2.3.40}$$

We define the vector $\vec{x} = \big(x_0, \ldots, x_N\big)^T \in \mathbb{Z}^N$ by

$$\vec{x} \,:=\, \mathcal{C}\vec{x}\,'\,. \tag{2.3.41}$$

Then we obtain

$$\big(0, \ldots, 0, e'\big)^T \,=\, \mathcal{D}\vec{x}\,' \,=\, \mathcal{A}\big(\mathcal{C}\vec{x}\,'\big) \,=\, \mathcal{A}\vec{x} \tag{2.3.42}$$

where

$$e' \,:=\, \sum_{\nu=0}^{N} e_\nu x'_\nu \,\ne\, 0\,. \tag{2.3.43}$$

Thus, the vector $\vec{x}$ represents an integer solution of the system (2.3.1) and (2.3.2). Now, we have with (2.3.40), (2.3.41) and with $N+1 \leqslant 2n$,

$$\begin{aligned} |\vec{x}| \;&=\; |\mathcal{C}\vec{x}\,'| \,\le\, (N+1)\cdot|\mathcal{C}|\cdot|\vec{x}\,'| \qquad\qquad (2.3.44)\\ &\leqslant\; (N+1)|\mathcal{C}|\big(c_5 e\sqrt{2}\big)^{M/2}\big(\log n\big)^{M/2} n^{M/2}|\mathcal{C}|^{M/2}\sqrt{D_1}\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^{M/2}\\ &\leqslant\; 2n|\mathcal{C}|^{1+M/2}\big(c_5 e\sqrt{2}\big)^{M/2}\big(\log n\big)^{M/2} n^{M/2}\sqrt{D_1}\big(\max\big\{D_1,\, c_5 n\log(n)\,|\mathcal{C}|\,\big\}\big)^{M/2}\,. \end{aligned}$$

This completes the proof of Lemma 8 for an integer solution $x_0, \ldots, x_N$ of the system in (2.3.9) and (2.3.10). ☐

2.4. **A series transformation for Ser's formula: the second fundamental lemma.**

In the following, let n be a power of two, say $n = 2^\alpha$ with $\alpha \geqslant 259$. Later, we will impose further conditions on α. We denote the set of prime numbers by $\mathbb{P}$. Furthermore, let

$$\beta \,:=\, \frac{1}{100} \,=\, 0.01\,, \qquad Q \,:=\, \lfloor n^\beta \rfloor\,, \qquad \tau \,:=\, \frac{1+Q}{3}\,. \tag{2.4.1}$$

A set P is defined by all integers between n/τ and $3n$ having at least one prime divisor in the same range:

$$P \,:=\, \big\{\, \nu p \big|\, p \in \mathbb{P} \wedge 1 \leqslant \nu \leqslant \min\{Q,\, \lfloor 3n/p \rfloor\} \wedge \nu \equiv 1 \pmod 2 \wedge n/\tau < p < 3n \big\}\,. \tag{2.4.2}$$

We require $Q \geqslant 6$, which implies $\tau \geqslant 7/3$. This is satisfied due to $\alpha \geqslant 259$. Then, the numbers k defined below by $qn \leqslant k \leqslant 3n$ with $k \notin P$ and $q = 45/32$ only have prime divisors $\leqslant n/\tau$. Let us assume that such a number k with $k \leqslant 3n$ has a prime divisor p with $n/\tau < p < 3n$. So, we have $k = mp$, say. By $m \leqslant 3n/p$, we even may assume that $m \leqslant \lfloor 3n/p \rfloor$. Since $k \notin P$, the remaining condition $\nu = m \leqslant Q$ cannot be satisfied; it follows that $m > Q$, or $m \geqslant Q+1$. Together with our assumption $p > n/\tau$, we have

$$k \,=\, mp \,\geqslant\, (Q+1)p \,>\, \frac{(Q+1)n}{\tau} \,=\, (Q+1)\frac{3n}{Q+1} \,=\, 3n\,,$$

a contradiction. The congruence $\nu \equiv 1 \pmod 2$ in (2.4.2) guarantees that the set P contains only odd numbers. Consequently, the two conditions $qn \leqslant k \leqslant 3n$ and $k \notin P$ do not exclude any even integer between qn and $3n$.

The following linear system in the unknowns x_0 and x_ν with $qn \leqslant k \leqslant 3n$ and $k \notin P$ will play a central role:

$$
\begin{aligned}
0 &= \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} \Big(x_k \sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + x_k \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} \Big) \Big) - x_0 \,, && (2.4.3) \\
0 &= \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} e(k,p) x_k \qquad \big(2 \leqslant p \leqslant n/\tau,\, p \in \mathbb{P} \big) \,, && (2.4.4) \\
0 &\neq r \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k - s x_0 \,, && (2.4.5)
\end{aligned}
$$

where r and s in (2.4.5) are natural numbers. While the linear forms on the right-hand sides of (2.4.4) and (2.4.5) already have integer coefficients, the same effect can be achieved for the form in (2.4.3) by multiplying by a common multiple V_n of the denominators.

The estimation of a suitable number V_n will be carried out later. First, the linear independence of the linear form defined on the right-hand side of (2.4.5) from the forms introduced on the right-hand sides of (2.4.3) and (2.4.4) is shown. In particular, we have $n \geqslant 512$. Moreover, we will use the symbol $\mu_{n,k}$, which is defined by

$$
\mu_{n,k} := \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} \Big) + \sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} \qquad \big(qn \leqslant k \leqslant 3n \big) \,. \tag{2.4.6}
$$

Furthermore, let $(p_\nu)_{\nu \geqslant 1}$ with $p_1 = 2$, $p_2 = 3, \ldots$ denote the sequence of all prime numbers. Assume that the linear form in (2.4.5) is linearly dependent over $\mathbb{Q}$ from the forms in (2.4.3) and (2.4.4). Then, there are rational numbers $\lambda_0, \lambda_1, \lambda_{p_1}, \lambda_{p_2} \ldots, \lambda_{p_{\pi(n/\tau)}}$ with

$$
\lambda_0 \Big(r \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k - s x_0 \Big) = \sum_{\nu=1}^{\pi(n/\tau)} \lambda_{p_\nu} \Big(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} e(k,p_\nu) x_k \Big) + \lambda_1 \Big(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} \mu_{n,k} x_k - x_0 \Big) \tag{2.4.7}
$$

and $\lambda_0 \neq 0$. By comparing coefficients at x_0, we obtain

$$
\lambda_1 = \lambda_0 s \,, \tag{2.4.8}
$$

and the comparison at x_k with $qn \leqslant k \leqslant 3n$ and $k \notin P$ results in the identities

$$
\lambda_0 r = \sum_{\nu=1}^{\pi(n/\tau)} \lambda_{p_\nu} e(k,p_\nu) + \lambda_1 \mu_{n,k} \qquad \big(qn \leqslant k \leqslant 3n,\, k \notin P \big) \,. \tag{2.4.9}
$$

We now write (2.4.9) explicitly again for four selected k. The following four numbers are divisible at most by the prime numbers $2, 3$ and 5 and therefore do not contain any prime divisors p with $p > n/\tau$, as occurs for numbers in the set P.

1.) $k = qn = 3^2 \cdot 5 \cdot 2^{\alpha-5} \notin P$. Note: $q = 45/32$.
Due to $e(k,p_\nu) = 0$ for $\nu > 3$ and $e(k,2) = \alpha - 5$, $e(k,3) = 2$ and $e(k,5) = 1$, (2.4.9)

takes the form

$$\lambda_0 r \;=\; \lambda_2(\alpha-5)+2\lambda_3+\lambda_5+\lambda_1\mu_{n,qn}\,. \tag{2.4.10}$$

2.) $k=3n/2=3\cdot 2^{\alpha-1}\notin P$. Note: $q<3/2<3$.
We have

$$\lambda_0 r \;=\; \lambda_2(\alpha-1)+\lambda_3+\lambda_1\mu_{n,3n/2}\,. \tag{2.4.11}$$

3.) $k=15n/8=3\cdot 5\cdot 2^{\alpha-3}\notin P$. Note: $q<15/8<3$.
Then, (2.4.9) gives

$$\lambda_0 r \;=\; \lambda_2(\alpha-3)+\lambda_3+\lambda_5+\lambda_1\mu_{n,15n/8}\,. \tag{2.4.12}$$

4.) $k=2n=2^{\alpha+1}\notin P$. Note: $q<2<3$.
We obtain

$$\lambda_0 r \;=\; \lambda_2(\alpha+1)+\lambda_1\mu_{n,2n}\,. \tag{2.4.13}$$

We subtract (2.4.10) from (2.4.11) and (2.4.12) from (2.4.13):

$$\left.\begin{aligned} 0 &= 4\lambda_2-\lambda_3-\lambda_5+\lambda_1\big(\mu_{n,3n/2}-\mu_{n,qn}\big)\,,\\ 0 &= 4\lambda_2-\lambda_3-\lambda_5+\lambda_1\big(\mu_{n,2n}-\mu_{n,15n/8}\big)\,. \end{aligned}\right\} \tag{2.4.14}$$

Due to $\lambda_0\neq 0$ and (2.4.8), λ_1 does not disappear. Therefore, from the two equations in (2.4.14), we obtain the identity $\mu_{n,2n}-\mu_{n,15n/8} \;=\; \mu_{n,3n/2}-\mu_{n,qn}$, or

$$\mu_{n,2n}-\mu_{n,3n/2} \;=\; \mu_{n,15n/8}-\mu_{n,qn}\,. \tag{2.4.15}$$

Now, due to Lemma 1, (2.1.23) and (2.4.6), we have

$$R_{n,k} \;=\; \gamma+\log k-\mu_{n,k}\,. \tag{2.4.16}$$

Using the two estimates from Lemma 4 and the identity in (2.4.16), we obtain an upper and a lower bound for $\mu_{n,k}$:

$$\gamma+\log k-\frac{1}{2k^2\binom{n+k}{k}} \;\leqslant\; \mu_{n,k} \;\leqslant\; \gamma+\log k-\frac{1}{7nk(k+1)\binom{n+k}{k}}\,. \tag{2.4.17}$$

This results, on the one hand, in

$$\mu_{n,15n/8}-\mu_{n,qn} \;\geqslant\; \log(4/3)+\frac{32}{15n^2}\cdot\Big(\frac{1}{21(45n/32+1)\binom{77n/32}{n}}-\frac{1}{15\binom{23n/8}{n}}\Big)\,, \tag{2.4.18}$$

on the other hand, in

$$\mu_{n,2n}-\mu_{n,3n/2} \;\leqslant\; \log(4/3)+\frac{1}{2n^2}\cdot\Big(\frac{4}{9\binom{5n/2}{n}}-\frac{1}{7(2n+1)\binom{3n}{n}}\Big)\,. \tag{2.4.19}$$

The two inequalities in (2.4.18) and (2.4.19) can be combined due to (2.4.15) to

$$\frac{32}{15n^2}\cdot\Big(\frac{32}{21(45n+32)\binom{77n/32}{n}}-\frac{1}{15\binom{23n/8}{n}}\Big) \;\leqslant\; \frac{1}{2n^2}\cdot\Big(\frac{4}{9\binom{5n/2}{n}}-\frac{1}{7(2n+1)\binom{3n}{n}}\Big)\,. \tag{2.4.20}$$

Here, we bring the terms in the two brackets to the lowest common denominator. Then we multiply the entire inequality by

$$3150n^2(2n+1)(45n+32)\binom{5n/2}{n}\binom{3n}{n}\binom{23n/8}{n}\binom{77n/32}{n}\,.$$

In this way, we finally obtain

$$\begin{aligned} & 25(45n+32)\binom{23n/8}{n}\binom{77n/32}{n}\left(28(2n+1)\binom{3n}{n}-9\binom{5/2n}{n}\right) \\ \geqslant\; & 64(2n+1)\binom{5n/2}{n}\binom{3n}{n}\left(160\binom{23n/8}{n}-7(45n+32)\binom{77n/32}{n}\right). \end{aligned} \tag{2.4.21}$$

We now refer to the two inequalities (2.2.9) and (2.2.10) in Lemma 7 and derive the following inequality for $n \geqslant 448$ with $n \equiv 0 \pmod{32}$:

$$\begin{aligned} & 700(45n+32)\binom{77n/32}{n}\binom{23n/8}{n} \\ \overset{(2.2.9)}{<}\; & 64\binom{5n/2}{n}\left(160\binom{23n/8}{n}-\binom{23n/8}{n}\right) \\ \overset{(2.2.10)}{<}\; & 64\binom{5n/2}{n}\left(160\binom{23n/8}{n}-7(45n+32)\binom{77n/32}{n}\right). \end{aligned} \tag{2.4.22}$$

From this, we obtain with (2.4.22):

$$\begin{aligned} & 64(2n+1)\binom{5n/2}{n}\binom{3n}{n}\left(160\binom{23n/8}{n}-7(45n+32)\binom{77n/32}{n}\right) \\ >\; & 25(45n+32)\binom{23n/8}{n}\binom{77n/32}{n}\cdot 28(2n+1)\binom{3n}{n} \\ >\; & 25(45n+32)\binom{23n/8}{n}\binom{77n/32}{n}\cdot\left(28(2n+1)\binom{3n}{n}-9\binom{5n/2}{n}\right). \end{aligned} \tag{2.4.23}$$

The inequalities (2.4.21) and (2.4.23) contradict each other. Therefore, the linear dependence of the linear form given on the right-hand side of (2.4.5) from the forms in (2.4.3) and (2.4.4) assumed in (2.4.7) cannot hold for $n = 2^\alpha$ with $\alpha \geqslant 259$.

In the next step of the series transformation, we first introduce a natural number V_n in such a way that the linear form from (2.4.3), multiplied by V_n,

$$V_n \cdot \left(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} \left(x_k \sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + x_k\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k-1}\right)\right) - x_0 \right), \tag{2.4.24}$$

becomes a linear form with integer coefficients. Here, V_n is the least common multiple of all denominators of the fractions appearing as coefficients of the x_k in (2.4.24). If all rational coefficients of x_k within the parentheses (including x_0) in (2.4.24) are reduced, then, after multiplication by V_n, all integer coefficients of the resulting linear form are relatively prime. Due to $qn \leqslant k \leqslant 3n$, we have

$$\frac{d(3n)}{k} \in \mathbb{N} \qquad \text{and} \qquad d(3n)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k-1}\right) \in \mathbb{N}. \tag{2.4.25}$$

Formula (2.2.2) in Lemma 5 with $a \to k$, $b \to n$ and $n_2 \to m$ guarantees for every k and every m with $0 \leqslant m \leqslant n$ that

$$\binom{k+m}{m} \,\Big|\, d(n)\binom{k+n}{n}. \tag{2.4.26}$$

Again, due to $qn \leqslant k \leqslant 3n$, a second application of (2.2.2) with $a \to k$, $b \to 3n$, and $n_2 \to k$ implies the arithmetic property

$$\binom{k+n}{n} \;=\; \binom{n+k}{k} \,\Big|\, d(3n)\binom{n+3n}{3n} \;=\; d(3n)\binom{4n}{n}. \tag{2.4.27}$$

If we multiply the linear form in (2.4.3) by

$$V_n' \;:=\; d(n)\big(d(3n)\big)^2\binom{4n}{n}, \tag{2.4.28}$$

due to (2.4.26) and (2.4.27), all binomial coefficients appear as divisors of V_n'. Additionally, due to (2.4.25), all isolated denominators $1, 2, \dots, k$ divide the second factor $d(3n)$ in V_n'. Now we must consider the denominators of the rational numbers t_{m+2} for $0 \leqslant m \leqslant n$. From (2.1.10) we can see that

$$(n+1)!d(n+2)\cdot t_{m+2} \;\in\; \mathbb{N} \qquad (0 \leqslant m \leqslant n). \tag{2.4.29}$$

Note that the polynomial to be integrated for t_{m+2} has degree $m+1$. Therefore, after multiplication by $d(m+2)$, an integer integral value is obtained. With $V_n'' := (n+1)!d(n+2)$, we now obtain a suitable number by $V_n'V_n''$ with $V_n|V_n'V_n''$ based on (2.4.28) and (2.4.29). In the following further estimation of V_n, the inequality (2.2.11) in Lemma 6 as well as Lemma 7 are applied. Therefore, let $n \geqslant 2^{1142} > 5\cdot 10^{343}$.

$$\begin{aligned}
V_n \;&\leqslant\; V_n'V_n'' \;=\; (n+1)!d(n)d(n+2)\big(d(3n)\big)^2\binom{4n}{n} && (2.4.30)\\
&\leqslant\; (n+1)^2(n+2)\big(d(n)d(3n)\big)^2\binom{4n}{n}n!\\
&<\; (n+1)^2(n+2)\big(2.72^n\cdot 2.72^{3n}\big)^2\Big(\frac{256}{27}\Big)^n n!\\
&=\; (n+1)^2(n+2)(2.72)^{8n}\Big(\frac{256}{27}\Big)^n n!\\
&<\; (n+1)^2(n+2)(28\,408)^n n!\,. && (2.4.31)
\end{aligned}$$

We now estimate $|\mu_{n,k}|$ roughly using (2.4.17). We obtain

$$2 \;<\; |\mu_{n,k}| \;<\; \gamma+\log k \;\leqslant\; \gamma+\log(3n) \;\leqslant\; n \qquad \big(n \geqslant 3,\, qn \leqslant k \leqslant 3n\big). \tag{2.4.32}$$

Due to (2.4.31), we have

$$\begin{aligned}
V_n|\mu_{n,k}| \;&<\; n(n+1)^2(n+2)(28\,408)^n n!\\
&<\; (28\,409)^n n! && (2.4.33)
\end{aligned}$$

for $n \geqslant 2^{1142}$ and $qn \leqslant k \leqslant 3n$. Three numerical examples show that one may conjecture the inequality

$$V_n \;>\; C^n n! \tag{2.4.34}$$

with a constant $C \leqslant 60$, but we do not need such a lower bound for V_n. The coefficients of the linear forms in (2.4.4) can be estimated much more easily: Every number k with $qn \leqslant k \leqslant 3n$ and $k \notin P$ can be factorized by its prime divisors,

$$k = \prod_{\nu=1}^{\pi(n/\tau)} p_\nu^{e(k,p_\nu)} \leqslant 3n . \tag{2.4.35}$$

Due to $p_\nu \geq 2$, we obtain by taking logarithms,

$$\log 2 \cdot \sum_{\nu=1}^{\pi(n/\tau)} e(k,p_\nu) \leqslant \log(3n) . \tag{2.4.36}$$

Because there is no negative summand on the left-hand side, it follows that

$$e(k,p_\nu) \leqslant \frac{\log(3n)}{\log 2} < 2\log n \qquad \bigl(n \geqslant 18,\, qn \leqslant k \leqslant 3n,\, k \notin P\bigr) . \tag{2.4.37}$$

Therefore, the constant c_3 in (2.3.7) can be set to $c_3 := 2$.

We summarize the results obtained so far in this subsection on linear independence and on (2.4.24), (2.4.33) and (2.4.37) in the following lemma.

Lemma 12. *(Second fundamental lemma)*
Let $n = 2^\alpha \geqslant 2^{1142}$.
(i) *With the number* V_n *defined above in (2.4.30), the linear form*

$$L_1 := \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} V_n \mu_{n,k} x_k - V_n x_0 \tag{2.4.38}$$

has integer coefficients $V_n\mu_{n,k}$*, which are uniformly bounded in* k *in terms of* n *by*

$$V_n|\mu_{n,k}| < (28\,409)^n n! \qquad \bigl(qn \leqslant k \leqslant 3n,\, k \notin P\bigr) . \tag{2.4.39}$$

(ii) *The coefficients of the linear forms*

$$L_p := \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} e(k,p) x_k \qquad \bigl(2 \leqslant p \leqslant n/\tau,\, p \in \mathbb{P}\bigr) \tag{2.4.40}$$

are uniformly bounded in k *and* p *in terms of* n *by*

$$e(k,p) < 2\log n \qquad \bigl(qn \leqslant k \leqslant 3n,\, k \notin P,\, 2 \leqslant p \leqslant n/\tau,\, p \in \mathbb{P}\bigr) . \tag{2.4.41}$$

(iii) *The linear form*

$$L := \Bigl(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} r x_k \Bigr) - s x_0 \tag{2.4.42}$$

is linearly independent over $\mathbb{Q}$ *from the linear forms in* $\bigl\{L_1\bigr\} \cup \bigl\{L_p : 2 \leqslant p \leqslant n/\tau,\, p \in \mathbb{P}\bigr\}$.

The system consisting of (2.4.3) to (2.4.5) can be written in abbreviated form as follows due to (2.4.38), (2.4.40) and (2.4.42):

$$\begin{aligned} L_1 &= 0\,, && (2.4.43)\\ L_p &= 0 \qquad \bigl(2 \leqslant p \leqslant n/\tau,\, p \in \mathbb{P}\bigr)\,, && (2.4.44)\\ L &\neq 0\,. && (2.4.45) \end{aligned}$$

This system is formed with $N+1$ variables from the set $\{x_0, x_{qn}, \ldots, x_{3n}\}$. In order to estimate N in both directions, we need an upper estimate for the number of elements in the set P. We obtain with the prime number theorem[12] and (4.3.7) for all powers of two $n = 2^\alpha$ (using $\alpha \geqslant 27\,411\,206$, $Q \leqslant n^\beta$, and $|P| > 1$ according to (4.3.10)):

$$\begin{aligned} 1 < |P| &= \Bigl|\bigl\{\, \nu p \bigr| p \in \mathbb{P} \wedge 1 \leqslant \nu \leqslant \min\{Q, \lfloor 3n/p \rfloor\} \wedge \nu \equiv 1 \pmod 2 \wedge n/\tau < p < 3n \bigr\}\Bigr| \\ &\leqslant \bigl|\,\bigl\{\, \nu p \bigr| p \in \mathbb{P} \wedge 1 \leqslant \nu \leqslant \min\{Q, \lfloor 3n/p \rfloor\} \wedge 2 \leqslant p < 3n \bigr\}\,\bigr| \\ &= \bigl|\,\bigl\{\, \nu p \bigr| p \in \mathbb{P} \wedge 2 \leqslant p \leqslant 3n/\nu \wedge 1 \leqslant \nu \leqslant Q \bigr\}\,\bigr| \\ &= \sum_{\nu=1}^{Q} \pi\Bigl(\frac{3n}{\nu}\Bigr) \leqslant \sum_{\nu=1}^{Q} \bigl(1+\varepsilon/2\bigr)\frac{3n}{\nu\log(3n/\nu)} \leqslant 3\bigl(1+\varepsilon/2\bigr)n \sum_{\nu=1}^{Q} \frac{1}{\nu\log(3n/n^\beta)} \\ &= \frac{3(1+\varepsilon/2)n}{\log(3n^{1-\beta})}\sum_{\nu=1}^{Q}\frac{1}{\nu} < \frac{3(1+\varepsilon/2)n}{\log(3n^{1-\beta})}\Bigl(\gamma + \frac{1}{Q} + \log Q\Bigr) \\ &< \frac{3(1+\varepsilon/2)n}{\log(3n^{1-\beta})}\bigl(\gamma + 1 + \log n^\beta\bigr) \overset{(4.3.7)}{<} \frac{3(1+\varepsilon)n}{\log(3n^{1-\beta})}\log n^\beta < \frac{3(1+\varepsilon)n}{\log(n^{1-\beta})}\log n^\beta \\ &= \frac{3\beta(1+\varepsilon)n}{1-\beta} < \frac{n}{32}\,. \end{aligned} \tag{2.4.46}$$

The latest estimate is based on $\beta = 1/100$. Therefore we obtain for

$$N := \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} 1 \; :$$

$$\frac{25n}{16} = \frac{50n}{32} < \bigl(3 - q - 1/32\bigr)n + 1 < \bigl(3n - qn + 1\bigr) - \bigl|P\,\bigr| = N < (3-q)n = \frac{51n}{32} \tag{2.4.47}$$

(note our condition $qn \in \mathbb{N}$). The left-hand inequality holds due to $3 - q - 1/32 = 50/32$, the right-hand inequality due to $|P| \geqslant 2$. In particular, (2.3.4) is satisfied in this way.

The number M of equations in (2.4.43) and (2.4.44) can be bounded as follows:

$$M := 1 + \pi(n/\tau) < 1 + (1+\varepsilon/2)\frac{n}{\tau\log(n/\tau)} < \frac{(1+\varepsilon)n}{\tau\log(n/\tau)}\,, \tag{2.4.48}$$

where $\varepsilon = 1/50\,000$ and $n = 2^\alpha$ is chosen sufficiently large using $\alpha \geqslant 218\,593$, see (4.3.6) in the Appendix. The right-hand inequality in (2.4.48) is equivalent to the inequality

$$\frac{2\tau\log(n/\tau)}{n} < \varepsilon\,,$$

---

[12] See (4.3.4) with $m = 3n$ in Section 4.3.

and applies to all natural numbers $n = 2^\alpha$ with $\alpha \geqslant 259$.

We add a preliminary consideration that we will need in the next section: For $n > 2^{100}$, we have

$$\tau \leqslant \frac{1}{3}\big(1 + n^{1/100}\big) < \frac{1}{2}n^{1/100}.$$

From this it follows that

$$\frac{n}{\tau} > \frac{2n}{n^{1/100}} = 2n^{0.99},$$

and in this way we have proven that

$$\log\left(\frac{n}{\tau}\right) > \log 2 + 0.99\log n > 0.99\log n.$$

The inequality often needed below is

$$\frac{1}{\log(n/\tau)} < \frac{1}{0.99\log n} = \frac{c_6}{\log n} \qquad (n > 2^{100}) \tag{2.4.49}$$

with $c_6 := 1/0.99$.

2.5. **A reduction problem for a lattice basis: the third fundamental lemma.**

In this section, we consider bases of the solution space of a linear Diophantine equation. This solution space is an $N+1$ dimensional lattice Λ. We are looking for a sublattice $\Lambda_{sub} \subseteq \Lambda$, which is spanned by $N+1$ vectors $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{N+1} \in \mathbb{Z}^{N+2}$. These vectors form a basis of the sublattice Λ_{sub}. We always assume that basis vectors are ordered in ascending order according to their maximum norm:

$$T_1 \leqslant T_2 \leqslant \ldots \leqslant T_{N+1}, \tag{2.5.1}$$

where

$$T_\mu := \big|\vec{t}_\mu\big| = \max_{1\leqslant\nu\leqslant N+2}\big|t_{\nu,\mu}\big| \qquad (1 \leqslant \mu \leqslant N+1). \tag{2.5.2}$$

The lattice constant of the lattice Λ is denoted by $d(\Lambda)$.

In this section, we shall state a third fundamental lemma, but before we need some preparing considerations.

Let us consider the linear Diophantine equation $L_1 = x_{N+1}$ given by (2.4.38) in the second fundamental lemma:

$$-V_n x_0 + \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} V_n \mu_{n,k} x_k - x_{N+1} = 0.$$

Based on the definition of N in (2.4.47), the sum on k can be rewritten as follows. Let $k_1 < k_2 < \ldots < k_N$ be the subscripts that satisfy simultaneously the conditions $qn \leqslant k \leqslant 3n$ and $k \notin P$:

$$-V_n x_0 + \sum_{\nu=1}^{N} V_n \mu_{n,k_\nu} x_\nu - x_{N+1} = 0. \tag{2.5.3}$$

Let

$$\ell_0 := V_n, \qquad l_0 := -V_n, \qquad l_\nu := V_n\mu_{n,k_\nu} = \ell_\nu \quad (1 \leqslant \nu \leqslant N). \tag{2.5.4}$$

We already know that the numbers $\ell_0, \ell_1, \ldots, \ell_N$ are coprime integers.

Lemma 13. *(Third fundamental lemma; Conjecture)*
Let $\varepsilon \geqslant 0$ *be a real number. For* $N > e^{81\,000}$ *and an integer* n *satisfying* $32N/51 < n < 16N/25$ *let* $\ell_0, \ell_1, \ldots, \ell_N > 0$ *be the numbers given by (2.5.4). We denote the solution basis of the linear Diophantine equation*

$$\ell_0 x_0 + \ell_1 x_1 + \cdots + \ell_N x_N - x_{N+1} \,=\, 0 \tag{2.5.5}$$

by Λ*. Then there exists an* $N+1$ *dimensional basis* $\{\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{N+1}\}$ *of a sublattice* $\Lambda_{sub} \subseteq \Lambda$*, where the lengths* $T_1, T_2, \ldots, T_{N+1}$ *defined by (2.5.2) and ordered by (2.5.1) are satisfying the following inequality*

$$T_{N+1}^{N/2} < \prod_{m=1}^{N+1} T_m \,<\, N^{2\varepsilon N/5} d(\Lambda)\,. \tag{2.5.6}$$

Remark 1. *The left-hand inequality in (2.5.6) follows from the condition*

$$T_1 \,>\, \sqrt{T_{N+1}}\,, \tag{2.5.7}$$

because we have, by $T_1 \leqslant T_2 \leqslant \ldots \leqslant T_{N+1}$*,*

$$T_{N+1}^{N/2} \overset{(2.5.7)}{<} T_1^N \,\leqslant\, T_1^{N+1} \,\leqslant\, T_1 T_2 \cdots T_{N+1}\,.$$

From the definition of $\ell_0, \ell_1, \ldots, \ell_N$ *in (2.5.4) we see that the third fundamental lemma can be reformulated as follows:*

> *For sufficiently large numbers* n*, the rationals of the set*
>
> $$\big\{\mu_{n,k} \,:\, qn \leqslant k \leqslant 3n \,\wedge\, k \notin P\big\}$$
>
> *generate an* ε*-balanced lattice. The success of the method chosen here to prove the irrationality of Euler's constant depends only on the validity of the inequality in (2.5.7)!*

We'll see in Section 3.1 below that Lemma 13 states the following, roughly speaking:

> If the solution space of the linear Diophantine equation has a basis consisting of vectors whose lengths differ only slightly from each other, then there is a basis of a sublattice consisting of relatively small solution vectors whose lengths do not differ too much from each other.

Of course, we'll apply this lemma with $\varepsilon = 1/50\,000$ and $n \geqslant 2^{27\,411\,206}$, so that the condition on N in the lemma holds by

$$N \,>\, \frac{25n}{16} \,=\, \frac{25}{16} \cdot 2^{\alpha} \,\geqslant\, \frac{25}{16} \cdot 2^{27\,411\,206} \,=\, 25 \cdot 2^{27\,411\,202} \,>\, e^{81\,000}\,.$$

The following two lemmas may contain preliminary properties for a proof of the third fundamental lemma. The first Lemma 14 provides an overview of the ratios of the coefficients of the linear equation (2.5.3) and thus of the vector norms of a non-reduced basis of the lattice Λ. The following Lemma 15 then shows that the signs of these coefficients do not play a significant role.

Lemma 14. *For $N > e^{81\,000}$ and an integer n satisfying $32N/51 < n < 16N/25$ we have the following inequalities for the numbers $\ell_0, \ell_1, \dots, \ell_N$:*

$$\ell_0 \,<\, \ell_1 < \ell_2 < \,\dots\, < \,\ell_N \,<\, n!(28\,409)^n\,, \tag{2.5.8}$$

$$0.45 + \log N \,<\, \frac{\ell_\nu}{\ell_0} \,<\, 1.23 + \log N \quad (1 \leqslant \nu \leqslant N)\,, \tag{2.5.9}$$

$$1 + \frac{0.499}{N\log N} \,<\, \frac{\ell_{\nu+1}}{\ell_\nu} \,<\, 1 + \frac{3.390}{N\log N} \quad (1 \leqslant \nu \leqslant N-1)\,. \tag{2.5.10}$$

Proof. Now, we are going to check whether the conditions (2.5.8) to (2.5.10) are satisfied for the linear Diophantine equation given by (2.5.3) and (2.5.4). We begin with a preliminary consideration. Let $l_{\nu_i} = V_n\mu_{n,k_i}$ for $i = 1,2$ with $1 \leqslant \nu_1 < \nu_2 \leqslant N$ and $k_i := k_{\nu_i}$ depending on ν_i by (2.5.4). Obviously, $qn \leqslant k_1 < k_2 \leqslant 3n$ is fulfilled.

The left-hand inequality in (2.5.8), $|l_0| < l_1$, is a consequence of (2.5.4) and (2.4.32), since $l_1 = V_n|\mu_{n,k}| > V_n = |l_0|$ holds for some k with $qn \leqslant k \leqslant 3n$ corresponding to l_1. To prove the remaining inequalities in (2.5.8), we fix two subscripts ν_1, ν_2 with $\nu_1 < \nu_2$ and consider the corresponding subscripts k_1, k_2 with $k_1 < k_2$. Then, we obtain from (2.4.17):

$$\begin{aligned}
\mu_{n,k_2} - \mu_{n,k_1} \;&\geqslant\; \left(\gamma + \log k_2 - \frac{1}{2k_2^2\binom{n+k_2}{k_2}}\right) - \left(\gamma + \log k_1 - \frac{1}{7nk_1(k_1+1)\binom{n+k_1}{k_1}}\right)\\
&=\; \log\left(\frac{k_2}{k_1}\right) + \frac{1}{7nk_1(k_1+1)\binom{n+k_1}{k_1}} - \frac{1}{2k_2^2\binom{n+k_2}{k_2}}\\
&>\; \log\left(\frac{k_2}{k_1}\right) - \frac{1}{2k_2^2}\,.
\end{aligned}$$

We now use the inequality $k_2 \geqslant k_1 + 1$. This leads to

$$\mu_{n,k_2} - \mu_{n,k_1} \,>\, \log\left(\frac{k_1+1}{k_1}\right) - \frac{1}{2(k_1+1)^2} \,=\, \log\left(1 + \frac{1}{k_1}\right) - \frac{1}{2(k_1+1)^2}\,.$$

From the power series expansion of the function $\log(1+x)$ with $0 < x \leqslant 1$, we obtain the inequality

$$\log(1+x) \,>\, x - \frac{x^2}{2} \qquad (0 < x \leqslant 1)\,. \tag{2.5.11}$$

Thus, we obtain

$$\begin{aligned}
\mu_{n,k_2} - \mu_{n,k_1} \;&>\; \frac{1}{k_1} - \frac{1}{2k_1^2} - \frac{1}{2(k_1+1)^2} \;=\; \frac{2k_1(k_1+1)^2 - (k_1+1)^2 - k_1^2}{2k_1^2(k_1+1)^2}\\
&=\; \frac{2k_1^3 + 2k_1^2 - 1}{2k_1^2(k_1+1)^2} \;>\; 0\,.
\end{aligned} \tag{2.5.12}$$

This gives

$$l_{\nu_1} \,=\, V_n\mu_{n,k_1} \,<\, V_n\mu_{n,k_2} \,=\, l_{\nu_2} \qquad (qn \leqslant k_1 < k_2 \leqslant 3n)\,, \tag{2.5.13}$$

which proves the desired inequalities $l_{\nu_1} < l_{\nu_2}$ for $1 \leqslant \nu_1 < \nu_2 \leqslant N$.

In the following, we will several times apply the two inequalities

$$\gamma - \frac{1}{64\,800} + \log k \;<\; \mu_{n,k} \;<\; \gamma + \log k \qquad (qn \leqslant k \leqslant 3n,\ k \notin P)\,, \tag{2.5.14}$$

which result directly from (2.4.17) for $n \geq 128$, since $k \geqslant qn = 45n/32 \geqslant 180$ and $2k^2 \geqslant 64\,800$. For a proof of the left-hand inequality in (2.5.9), we use the middle relation from (2.5.14) with some $qn \leqslant k \leqslant 3n$ corresponding to l_ν, whereas for the right-hand inequality in (2.5.9) the right-hand relation in (2.5.14) will be applied:

$$\begin{aligned}\frac{l_\nu}{|l_0|} \;&=\; \frac{V_n\mu_{n,k}}{V_n} = \mu_{n,k} \;>\; \gamma - \frac{1}{64\,800} + \log(qn) \;>\; \gamma - \frac{1}{64\,800} + \log\Big(\frac{45\cdot 32}{32\cdot 51}\Big) + \log N \\ &>\; 0.45 + \log N\,,\end{aligned}$$

note that $q = 45/32$ and $n > 32N/51$ by (2.4.47). Next, we have

$$\frac{l_\nu}{|l_0|} \;<\; \gamma + \log(3n) \;<\; \gamma + \log 3 + \log(16/25) + \log N \;<\; 1.23 + \log N\,.$$

Here we have used $n < 16N/25$.

To prove the left-hand inequality in (2.5.10), we refer once again to the inequality in (2.5.12) and replace the lower estimate 0 by a slightly better bound:

$$\frac{2k_1^3 + 2k_1^2 - 1}{2k_1^2(k_1+1)^2} \;>\; \frac{1}{k_1+2}\,.$$

This inequality is equivalent with $2k_1^3 + 2k_1^2 - k_1 - 2 > 0$ and holds for $k_1 \geqslant 1$. Hence, we have

$$\mu_{n,k_2} - \mu_{n,k_1} \overset{(2.5.12)}{>} \frac{1}{k_1+2}\,,$$

and, consequently, by using $k_1 \leqslant 3n < 48N/25$,

$$\begin{aligned}\frac{\mu_{n,k_2}}{\mu_{n,k_1}} \;&>\; 1 + \frac{1}{(k_1+2)\mu_{n,k_1}} \overset{(2.5.14)}{>} 1 + \frac{1}{(k_1+2)(\gamma + \log k_1)} \\ &>\; 1 + \frac{25}{(50+48N)(\gamma + \log(48/25) + \log N)} \\ &>\; 1 + \frac{12}{(25+24N)\log N} \;=\; 1 + \frac{12}{(24+25/N)N\log N}\,.\end{aligned} \tag{2.5.15}$$

The final estimate in (2.5.15) follows from the equivalent relation

$$\log N \;>\; 24\Big(\gamma + \log\frac{48}{25}\Big)\,,$$

which is fulfilled for $N > 65\cdot 10^{11}$. This size of N also guarantees

$$\frac{12}{24 + 25/N} \;>\; 0.499\,,$$

and so we finally obtain from (2.5.15),

$$\frac{\mu_{n,k_2}}{\mu_{n,k_1}} \;>\; 1 + \frac{0.499}{N\log N}\,,$$

which proves the left-hand inequality in (2.5.10) by making use of the two equations in (2.5.13).

For a proof of the right-hand inequality in (2.5.10), we again apply (2.4.17):

$$\begin{aligned}
\mu_{n,k_2} - \mu_{n,k_1} &\leqslant \left(\gamma + \log k_2 - \frac{1}{7nk_2(k_2+1)\binom{n+k_2}{k_2}} \right) - \left(\gamma + \log k_1 - \frac{1}{2k_1^2\binom{n+k_1}{k_1}} \right) \\
&= \log\left(\frac{k_2}{k_1}\right) + \frac{1}{2k_1^2\binom{n+k_1}{k_1}} - \frac{1}{7nk_2(k_2+1)\binom{n+k_2}{k_2}} \\
&< \log\left(\frac{k_2}{k_1}\right) + \frac{1}{k_1}\,.
\end{aligned}$$

Let ψ be an integer given by

$$\psi := \max_{\substack{qn \leqslant k_1 < k_2 \leqslant 3n \\ k_1, k_2 \notin P}} \big(k_2 - k_1\big)\,,$$

where k_1 and k_2 belong to subsequent numbers l_ν and $l_{\nu+1}$, respectively. We already know from the definition of the set P in (2.4.2) that k takes at least every even number when running through the interval between qn and $3n$. Therefore, using $k_2 \leqslant k_1+\psi$ with $1 \leqslant \psi \leqslant 2$ and the fact that k_1, k_2 belong to subsequent numbers in $\{qn, qn+1, \dots, 3n\} \setminus P$ because l_ν and $l_{\nu+1}$ denote subsequent numbers, we obtain

$$\begin{aligned}
\mu_{n,k_2} - \mu_{n,k_1} &< \log\left(\frac{k_1+\psi}{k_1}\right) + \frac{1}{k_1} \leqslant \log\left(1+\frac{2}{k_1}\right) + \frac{1}{k_1} \\
&< \frac{2}{k_1} + \frac{1}{k_1} \leqslant \frac{3}{qn} = \frac{32\cdot 3}{45n} \leqslant \frac{51\cdot 3}{45N} = \frac{17}{5N}\,.
\end{aligned}$$

We divide this inequality with μ_{n,k_1}. This yields, together with the lower bound in (2.5.14),

$$\begin{aligned}
\frac{l_{\nu+1}}{l_\nu} &= \frac{\mu_{n,k_2}}{\mu_{n,k_1}} < 1 + \frac{17}{5\mu_{n,k_1}N} \\
&< 1 + \frac{17}{5(\gamma - 1/64\,800 + \log k_1)N} \\
&< 1 + \frac{17}{5(\gamma - 1/64\,800 + \log(\frac{45}{32}\frac{32}{51}) + \log N)N}\,.
\end{aligned}$$

We are only setting temporarily,

$$\begin{aligned}
A &:= \frac{17}{5}\,, \\
B &:= \gamma - \frac{1}{64\,800} + \log\left(\frac{15}{17}\right), \\
C &:= 3.39\,.
\end{aligned}$$

Then, the inequality

$$\frac{A}{B+\log N} < \frac{C}{\log N}$$

holds, if

$$\log N \;>\; \frac{BC}{A-C} \;=\; 153.2405\ldots\,.$$

So, the inequality

$$\frac{l_{\nu+1}}{l_\nu} \;<\; \frac{3.39}{N\log N}$$

is fulfilled for $N > e^{154}$, which is guaranteed by the hypothesis of Lemma 14.
Note that in the above proof we have used $n > 32N/51$ according to (2.4.47). All conditions of Lemma 14 are therefore satisfied for the Diophantine equation in (2.5.3). □

Lemma 15. *The third fundamental lemma remains true, when the numbers* $\ell_0, \ell_1, \ldots, \ell_N$ *are replaced by* $l_0 = \pm\ell_0, l_1 = \pm\ell_1, \ldots, l_N = \pm\ell_N$*.*

Proof. Obviously, we have

$$|l_0| \;<\; |l_1| < |l_2| < \;\ldots\; < \;|l_N| \;<\; n!(28\,409)^n\,, \tag{2.5.16}$$

and the bounds given in (2.5.9) hold for $|l_\nu|/|l_0|$, while the bounds in (2.5.10) hold for $|l_{\nu+1}|/|l_\nu|$.
First, we apply the third fundamental lemma to the Diophantine equation

$$|l_0|x_0 + |l_1|x_1 + \;\ldots\; + |l_N|x_N - x_{N+1} \;=\; 0\,. \tag{2.5.17}$$

By (2.5.16) and the fundamental lemma, there exists a basis $\{\vec{t_1}, \vec{t_2}, \ldots, \vec{t}_{N+1}\}$ of a sublattice $\Lambda_{sub} \subseteq \Lambda$ satisfying the inequalities and conditions in (2.5.6). In particular, the vectors $\vec{t_1}, \vec{t_2}, \ldots, \vec{t}_{N+1}$ are linearly independent over $\mathbb{Q}$: The identity

$$q_1\vec{t_1} + q_2\vec{t_2} + \;\ldots\; + q_{N+1}\vec{t}_{N+1} \;=\; \vec{0}\,, \tag{2.5.18}$$

where $q_1, q_2, \ldots, q_{N+1}$ are rational unknown numbers, is equivalent with a linear system of $N+2$ equations in $N+1$ unknowns,

$$q_1 t_{\nu,1} + q_2 t_{\nu,2} + \;\ldots\; + q_{N+1} t_{\nu,N+1} \;=\; 0 \qquad (\nu = 1, 2, \ldots, N+2)\,. \tag{2.5.19}$$

Here, the numbers $t_{\nu,\mu}$ are the components of the vectors

$$\vec{t_\mu} \;=\; \begin{pmatrix} t_{1,\mu} \\ t_{2,\mu} \\ \vdots \\ t_{N+2,\mu} \end{pmatrix} \qquad (\mu = 1, 2, \ldots, N+1)\,, \tag{2.5.20}$$

and we have the equation

$$\sum_{\nu=0}^{N} \bigl|l_\nu\bigr| \cdot t_{\nu+1,\mu} - t_{N+2,\mu} \;=\; 0 \tag{2.5.21}$$

by definition of the vectors $\vec{t_\mu}$. Now, (2.5.18) holds only for

$$q_1 \;=\; q_2 \;=\; \ldots \;=\; q_{N+1} \;=\; 0\,, \tag{2.5.22}$$

since $\vec{t_1}, \vec{t_2}, \ldots, \vec{t}_{N+1}$ are linearly independent over $\mathbb{Q}$.

Next, we consider the Diophantine equation

$$l_0x_0 + l_1x_1 + \;\ldots\; + l_Nx_N - x_{N+1} \;=\; 0\,. \tag{2.5.23}$$

A basis for a sublattice of the general solution lattice of (2.5.23) is given by the set of vectors $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{N+1}\}$, where

$$\vec{w}_\mu := \begin{pmatrix} \operatorname{sgn}(l_0)t_{1,\mu} \\ \operatorname{sgn}(l_1)t_{2,\mu} \\ \vdots \\ \operatorname{sgn}(l_N)t_{N+1,\mu} \\ t_{N+2,\mu} \end{pmatrix} \qquad (\mu = 1, 2, \ldots, N+1)\,. \tag{2.5.24}$$

(i) The components of every vector $\vec{w}_\mu$ solve the equation in (2.5.23), because

$$\bigl|l_\nu\bigr| = \operatorname{sgn}(l_\nu) \cdot l_\nu \qquad (\nu = 0, 1, \ldots, N) \tag{2.5.25}$$

and

$$\begin{aligned} \sum_{\nu=0}^{N} l_\nu\bigl(\operatorname{sgn}(l_\nu) \cdot t_{\nu+1,\mu}\bigr) - t_{N+2} &= \sum_{\nu=0}^{N} \bigl(l_\nu \cdot \operatorname{sgn}(l_\nu)\bigr)t_{\nu+1,\mu} - t_{N+2} \\ \overset{(2.5.25)}{=} \sum_{\nu=0}^{N} \bigl|l_\nu\bigr| \cdot t_{\nu+1,\mu} - t_{N+2} \overset{(2.5.21)}{=} 0 \end{aligned}$$

holds for every $\mu = 1, 2, \ldots, N+1$.

(ii) The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{N+1}$ are linearly independent over $\mathbb{Q}$: Let $q_1^*, q_2^*, \ldots, q_{N+1}^*$ be rational numbers satisfying

$$q_1^*\vec{w}_1 + q_2^*\vec{w}_2 + \ \ldots\ + q_{N+1}^*\vec{w}_{N+1} = \vec{0}\,.$$

Using (2.5.24), this is equivalent with the system of equations,

$$\left.\begin{array}{rcl} q_1^* \operatorname{sgn}(l_{\nu-1})t_{\nu,1} + q_2^* \operatorname{sgn}(l_{\nu-1})t_{\nu,2} + \ \ldots\ + q_{N+1}^* \operatorname{sgn}(l_{\nu-1})t_{\nu,N+1} & = & 0 \\ q_1^*t_{N+2,1} + q_2^*t_{N+2,2} + \ \ldots\ + q_{N+1}^*t_{N+2,N+1} & = & 0 \end{array}\right\} \tag{2.5.26}$$

for every subscript $\nu = 1, 2, \ldots, N+1$. Since no coefficient $l_0, l_1, \ldots, l_N$ vanishes by the hypothesis of the lemma, the system in (2.5.26) is again equivalent with the system

$$q_1^*t_{\nu,1} + q_2^*t_{\nu,2} + \ \ldots\ + q_{N+1}^*t_{\nu,N+1} = 0 \qquad (\nu = 1, 2, \ldots N+2)\,.$$

Taking (2.5.19) and (2.5.22) into account, we conclude on $q_1^* = q_2^* = \ldots = q_{N+1}^* = 0$.

(iii) The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{N+1}$ satisfy the conditions in (2.5.6), because

$$T_\mu^* := \|\vec{w}_\mu\|_\infty \overset{(2.5.24)}{=} \|\vec{t}_\mu\|_\infty = T_\mu \qquad (\mu = 1, 2, \ldots, N+1)\,.$$

This completes the proof of the lemma. □

3. ATTEMPT OF A PROOF OF CONJECTURE 1.

3.1. **A matrix $\mathcal{C}$ for the application of the first fundamental lemma: a base reduction with the third fundamental lemma.**

In order to construct a suitable matrix $\mathcal{C}$ for the application of the first fundamental lemma, we apply the third fundamental lemma to the linear Diophantine equation $L_1 = x_{N+1}$ given by (2.4.38). Then, by Lemmas 13 and 15 (with $-\ell_0$ instead of ℓ_0), there exists a base $\{\vec{t}_1, \vec{t}_2, \dots, \vec{t}_{N+1}\}$ of the solution lattice of the linear Diophantine equation (2.5.3), where the lengths $T_1 \leqslant \dots \leqslant T_{N+1}$ of the base vectors satisfy the conditions in (2.5.6).

1.) Next, due to inequality (2.5.6), we have (recall the definition of $T_\mu = |\vec{t}_\mu|$ in (2.5.2) and the order of magnitude in (2.5.1)):

$$
\begin{aligned}
\prod_{m=1}^{N+1} T_m \quad & < \quad N^{2\varepsilon N/5} d(\Lambda) \;=\; N^{2\varepsilon N/5}\sqrt{1 + l_0^2 + l_1^2 + \ldots + l_N^2} \\
& \leqslant \quad N^{2\varepsilon N/5}\sqrt{(N+2)\max\{1, \ell_0^2, \dots, \ell_N^2\}} \\
& \leqslant \quad N^{2\varepsilon N/5}\sqrt{N+2}\cdot\sqrt{V_n^2 \max_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} \{1, \mu_{n,k}^2\}} \\
& \overset{(2.5.14)}{<} \quad N^{2\varepsilon N/5} V_n \sqrt{N+2} \max_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} \{1, \gamma + \log k\} \\
& < \quad N^{2\varepsilon N/5} V_n \sqrt{N+2}\cdot\bigl(\gamma + \log(3n)\bigr) \\
& \overset{(2.4.39)}{<} \quad \sqrt{2n}\bigl(\gamma + \log(3n)\bigr)(2n)^{51\varepsilon n/80}(28\,409)^n n! \\
& < \quad \sqrt{2n}\bigl(\gamma + \log(3n)\bigr)(28\,410)^n n^{51\varepsilon n/80} n! \\
& < \quad n^{2\varepsilon n/3} n!\,.
\end{aligned}
\tag{3.1.1}
$$

Here, we have used the following facts:

$$
\begin{aligned}
N \quad & < \quad 51n/32 < 2n \qquad (n \geqslant 1)\,, \\
N + 2 \quad & < \quad 51n/32 + 2 < 2n \quad (n \geqslant 5)\,, \\
2^{51\varepsilon/80}\cdot 28\,409 \quad & < \quad 28\,410\,, \quad (\text{since } \varepsilon = 1/50\,000)\,; \\
\sqrt{2n}\bigl(\gamma + \log(3n)\bigr)(28\,410)^n n^{51\varepsilon n/80} \quad & < \quad n^{2\varepsilon n/3} \quad (n = 2^\alpha \text{ mit } \alpha \geqslant 27\,411\,206)\,.
\end{aligned}
$$

The last inequality is proven in the Appendix in Section 4.3.

2.) To construct a regular matrix $\mathcal{C}$ required for the transformation in the first fundamental lemma, we first prove the following auxiliary theorem, which is a simple consequence of the equality of row and column rank of a matrix.

Lemma 16. *Let $N > 0$ be a natural number. Furthermore, let*

$$
L\bigl(x_0, \dots, x_N\bigr) \;:=\; \sum_{\nu=0}^{N} \ell_\nu x_\nu \tag{3.1.2}
$$

be a linear form with non-zero integer coefficients $\ell_0, \ldots, \ell_N$. *Furthermore, let*

$$\mathcal{C}' := \begin{pmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,N} & c_{0,N+1} \\ c_{1,1} & c_{1,2} & \cdots & c_{1,N} & c_{1,N+1} \\ \vdots & & \vdots & & \vdots \\ c_{N,1} & c_{N,2} & \cdots & c_{N,N} & c_{N,N+1} \\ c_1' & c_2' & \cdots & c_N' & c_{N+1}' \end{pmatrix} \in \mathbb{Z}^{(N+2)\times(N+1)} \tag{3.1.3}$$

be a matrix such that the properties

$$rg\big(\mathcal{C}'\big) = N+1 \tag{3.1.4}$$

and

$$c_\mu' := L\big(c_{0,\mu}, c_{1,\mu}, \ldots, c_{N,\mu}\big) = \sum_{\nu=0}^{N} \ell_\nu c_{\nu,\mu} \qquad (1 \leqslant \mu \leqslant N+1). \tag{3.1.5}$$

are fulfilled. Then the quadratic $(N+1)\times(N+1)$ *matrix* $\mathcal{C}$*, given by*

$$\mathcal{C} := \begin{pmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,N} & c_{0,N+1} \\ c_{1,1} & c_{1,2} & \cdots & c_{1,N} & c_{1,N+1} \\ \vdots & & \vdots & & \vdots \\ c_{N,1} & c_{N,2} & \cdots & c_{N,N} & c_{N,N+1} \end{pmatrix}, \tag{3.1.6}$$

has full rank, and its column vectors satisfy the inequalities

$$\Big|\sum_{\nu=0}^{N} \ell_\nu c_{\nu,\mu}\Big| \leqslant \max\big\{|c_1'|, |c_2'|, \ldots, |c_{N+1}'|\big\} \qquad (1 \leqslant \mu \leqslant N+1). \tag{3.1.7}$$

Due to $\ell_0\ell_1\cdots\ell_N \neq 0$, the column vectors of the matrix $\mathcal{C}'$ thus form a basis of the $N+1$ dimensional solution space of the Diophantine equation $L(x_0,\ldots,x_N) - y = 0$ with $N+2$ unknowns $x_0,\ldots,x_N,y$. With its full rank, the quadratic matrix $\mathcal{C}$ in (3.1.6) is regular. It will be identified later with the matrix $\mathcal{C}$ in Lemma 8 and Lemma 9; cf. (2.3.13).

Proof. The inequalities in (3.1.7) result directly from the identities assumed in (3.1.5). It remains to show that $\mathrm{rg}(\mathcal{C}) = N+1$ holds for the matrix $\mathcal{C}$ defined in (3.1.6). From (3.1.5) it follows that

$$\begin{pmatrix} c_1' \\ c_2' \\ \vdots \\ c_{N+1}' \end{pmatrix} = \begin{pmatrix} \ell_0 c_{0,1} + \ell_1 c_{1,1} + \cdots + \ell_N c_{N,1} \\ \ell_0 c_{0,2} + \ell_1 c_{1,2} + \cdots + \ell_N c_{N,2} \\ \vdots \\ \ell_0 c_{0,N+1} + \ell_1 c_{1,N+1} + \cdots + \ell_N c_{N,N+1} \end{pmatrix} = \mathcal{C}^T \begin{pmatrix} \ell_0 \\ \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}. \tag{3.1.8}$$

Therefore, the last row in the matrix $\mathcal{C}'$ is a linear combination of the remaining rows of the same matrix. However, since in $\mathcal{C}'$ the rank is both equal to the row rank and the column rank, and since due to (3.1.4) $N+1$ rows are already linearly independent, the desired rank statement $\mathrm{rg}(\mathcal{C}) = \mathrm{rg}(\mathcal{C}') = N+1$ follows from (3.1.8). □

With the basis vectors $\{\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{N+1}\}$ of a sublattice Λ_{sub} given by Lemma 14, we construct the matrix

$$\mathcal{C}' := (\vec{t}_1^{\,T} \ldots \vec{t}_{N+1}^{\,T}) \in \mathbb{Z}^{(N+2)\times(N+1)}.$$

It has rank $N+1$ because its column vectors form a basis of a subspace for the solution space of the equation in (2.5.5) and they are therefore linearly independent over $\mathbb{Q}$. Let

$$\vec{t}_\nu \;=:\; \bigl(t_{0,\nu}, t_{1,\nu}, \ldots, t_{N,\nu}, t_{N+1,\nu}\bigr) \;\in\; \mathbb{Z}^{N+2} \qquad (1 \leqslant \nu \leqslant N+1) \tag{3.1.9}$$

and

$$\mathcal{C} \;:=\; \begin{pmatrix} t_{0,1} & \ldots & t_{0,N+1} \\ t_{1,1} & \ldots & t_{1,N+1} \\ \vdots & & \vdots \\ t_{N,1} & \ldots & t_{N,N+1} \end{pmatrix} \;\in\; \mathbb{Z}^{(N+1)\times(N+1)}\,. \tag{3.1.10}$$

The vector $\vec{a}_1 = (a_{1,0}, a_{1,1}, \ldots, a_{1,N})$ from the system of equations considered in Lemma 8 is now specified by $\vec{a}_1 = (l_0, l_1, \ldots, l_N) = V_n(\mu_{n,k_0}, \mu_{n,k_1}, \ldots, \mu_{n,k_{N+1}})$. It follows that

$$\vec{a}_1\mathcal{C} \;=\; \bigl(l_0, l_1, \ldots, l_N\bigr)\mathcal{C} \;\overset{(2.5.5)}{=}\; \bigl(t_{N,1}, t_{N,2}, \ldots, t_{N,N+1}\bigr)\,. \tag{3.1.11}$$

Then, we apply Lemma 16 by setting

$$\begin{aligned} \ell_\nu \;&:=\; l_\nu\,, \\ c_{\mu,\nu} \;&:=\; t_{\mu,\nu}\,, \\ c'_\nu \;&:=\; t_{N+1,\nu} \end{aligned}$$

for $1 \leqslant \nu \leqslant N+1$ and $0 \leqslant \mu \leqslant N$. We collect our preceding results and obtain from Lemma 16, (3.1.10) and (3.1.11),

$$\mathrm{rg}\bigl(\mathcal{C}\bigr) \;=\; N+1\,, \quad |\mathcal{C}| \;\leqslant\; T_{N+1} \quad \text{and} \quad |\vec{a}_1\mathcal{C}| \;\leqslant\; T_{N+1}\,, \tag{3.1.12}$$

3.) We know from the left-hand side of (2.5.6) in Lemma 13 that

$$T_{N+1} \;\leqslant\; \Bigl(\,\prod_{\mu=1}^{N+1} T_\mu\Bigr)^{\beta_4} \tag{3.1.13}$$

with

$$\beta_4 \;:=\; \frac{2}{N}\,. \tag{3.1.14}$$

Next, let

$$\left.\begin{aligned} \beta_1 \;&:=\; \tau - 2\varepsilon\,, \\ \beta_2 \;&:=\; \tau - 1 - \varepsilon \end{aligned}\right\} \tag{3.1.15}$$

and

$$\beta_3 \;:=\; 2\,, \tag{3.1.16}$$

so that $\beta_4 = \beta_3/N$. For β_1 and β_2, we have

$$1 \;<\; \beta_1 \;<\; 1 + \beta_2 \;<\; \tau\,; \tag{3.1.17}$$

the inequality $\beta_1 > 1$ follows from $\tau \geqslant 7/3$ due to $Q \geqslant 6$ and $\varepsilon = 1/50\,000$. Moreover, the inequality

$$\beta_3 \;=\; 2 \;<\; \frac{25}{16}\Bigl(\,\min\bigl\{\beta_1, \beta_2\bigr\} - \varepsilon\,\Bigr) \;=\; \frac{25}{16}\bigl(\,\beta_2 - \varepsilon\,\bigr) \;=\; \frac{25}{16}\bigl(\,\tau - 1 - 2\varepsilon\,\bigr) \tag{3.1.18}$$

holds, since

$$\frac{2\cdot 16}{25} \;<\; \frac{7}{3} - 1 - 2\varepsilon\,.$$

Using (3.1.1) and the upper bound for T_{N+1} derived in (3.1.13), we obtain:

$$T_{N+1} \leqslant \Big(\prod_{\mu=1}^{N+1} T_\mu \Big)^{\beta_4} \leqslant \big(n^{2\varepsilon n/3} n! \big)^{\beta_4} . \tag{3.1.19}$$

Starting with (3.1.14) and $\beta_4 = \beta_3/N$, it follows that

$$\begin{aligned} T_{N+1} &< \big(n^{2\varepsilon n/3} n! \big)^{\beta_3/N} \overset{(2.4.47)}{<} \big(n^{2\varepsilon n/3} n! \big)^{16\beta_3/25n} \\ &\overset{(2.2.12)}{<} \Big(n^{2\varepsilon n/3} \sqrt{2\pi(n+1)} e^{-n} n^n \Big)^{16\beta_3/25n} < \big(n^{2\varepsilon n/3} n^n \big)^{16\beta_3/25n} \\ &= n^{32\varepsilon\beta_3/75} n^{16\beta_3/25} < n^{\varepsilon} n^{16\beta_3/25} \\ &= n^{\varepsilon + 16\beta_3/25} \overset{(3.1.18)}{<} n^{\min\{\beta_1,\beta_2\}} . \end{aligned} \tag{3.1.20}$$

Here, we have used:

$$\begin{aligned} \sqrt{2\pi(n+1)} e^{-n} &< 1 \quad (n \geqslant 2) , \\ \frac{32\beta_3}{75} &\overset{(3.1.16)}{=} \frac{32 \cdot 2}{75} = \frac{64}{75} < 1 . \end{aligned}$$

Therefore, we obtain the final result of this section by combining (3.1.12) and (3.1.20) for sufficiently large integers n:

$$\operatorname{rg}\big(\mathcal{C}\big) = N+1 , \quad |\mathcal{C}| < n^{\beta_2} \quad \text{and} \quad D_1 = |\vec{a}_1 \mathcal{C}| < n^{\beta_1} , \tag{3.1.21}$$

where the matrix $\mathcal{C}$ is given by (3.1.10).

3.2. **An approximate construction based on the first and second fundamental lemma.**

We now specify the system in (2.3.1) and (2.3.2) as follows:

$$\begin{aligned} 0 &= \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} V_n \Big(x_k \sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + x_k \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} \Big) \Big) - V_n x_0 \\ &= \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} V_n \mu_{n,k} x_k - V_n x_0 \end{aligned} \tag{3.2.1}$$

$$0 = \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} e(k,p) x_k \qquad \big(2 \leqslant p \leqslant \frac{n}{\tau}, \, p \in \mathbb{P} \big) \tag{3.2.2}$$

$$0 \neq \Big(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} r x_k \Big) - s x_0 . \tag{3.2.3}$$

Here, r and s in (3.2.3) are arbitrarily chosen positive integers. Then, all the conditions in (2.3.4) to (2.3.8) are fulfilled by Lemma 12 including the linear independence of the linear form on the right-hand side in (3.2.3) from the linear forms in (3.2.1) and (3.2.2). The numerical values of the constants c_2 and c_3 are listed in the Appendix in Section 4.4. Therefore, using the matrix $\mathcal{C}$ constructed in Section 3.1, we may apply Lemma 8. For this purpose, we still have to check the condition $\omega_\mu \leqslant c_4 N/\mu$ for $2 \leqslant \mu \leqslant M$. Note that we denote the sequence of primes by $(p_j)_{j\geqslant 1} = (2, 3, 5, 7, 11, \ldots)$. Then, we obtain

$$\begin{aligned}
\omega_\mu &= \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P \\ e(k,p_{\mu-1}) > 0}} 1 \leqslant \sum_{\substack{qn \leqslant k \leqslant 3n \\ e(k,p_{\mu-1}) > 0}} 1 = \sum_{\substack{qn \leqslant k \leqslant 3n \\ p_{\mu-1} | k}} 1 \\
&\leqslant \sum_{\substack{2 \leqslant k \leqslant 3n \\ p_{\mu-1} | k}} 1 = \left\lfloor \frac{3n}{p_{\mu-1}} \right\rfloor \\
&\leqslant \frac{3n}{p_{\mu-1}} \leqslant \frac{3n}{\mu-1} \overset{(\mu\geqslant 2)}{\leqslant} \frac{6n}{\mu} \\
&\overset{(2.4.47)}{\leqslant} \frac{6 \cdot 16N}{25\mu} = \frac{96N}{25\mu}\,.
\end{aligned}$$

It turns out that the condition for ω_μ holds with the constant $c_4 := 96/25$.

In this way, we can now apply inequality (2.3.11) from the first fundamental lemma to estimate the solution components of a special solution of the system in (3.2.1) to (3.2.3). Here, we have $c_5 = 384/25$. This bound will now be further enlarged in the following, with the new bound being expressed exclusively in terms of n, β_1, β_2 and τ. With $\varepsilon = 1/50\,000$, we now choose $n = 2^\alpha$ large enough so that the inequality

$$\frac{\log(c_5^{1/\tau} e^{1/2\tau} 2^{1/4\tau})}{\log n} + \frac{\log\log n}{\tau \log n} < \frac{\varepsilon}{1+\varepsilon} < \frac{\big(2 - (1+\beta_2)/\tau\big)\varepsilon}{1+\varepsilon} \tag{3.2.4}$$

is also satisfied. Using a computer, we find that the left-hand inequality in (3.2.4) holds for $n = 2^\alpha$ with $\alpha \geqslant 500\,000$: see 3.) in Appendix 4.3. The right-hand inequality follows simply by $1 < 2 - (1+\beta_2)/\tau$. The inequality in (3.2.4) can be rearranged equivalently as follows:

$$\frac{(1+\varepsilon)n}{\log n} \cdot \log\left(c_5^{1/\tau} e^{1/2\tau} 2^{1/4\tau} \log^{1/\tau} n\right) + \frac{1+\beta_2}{\tau}\varepsilon n < 2\varepsilon n\,.$$

Together with $(1+\beta_2)/\tau < 1$, this gives

$$\left(c_5^{1/\tau} e^{1/2\tau} 2^{1/4\tau} \log^{1/\tau} n\right)^{(1+\varepsilon)c_6 n/\log n} \cdot e^{\varepsilon c_6 n (1+\beta_2)/\tau} < e^{2\varepsilon c_6 n}\,, \tag{3.2.5}$$

where $c_6 > 0$ is a constant defined later. Furthermore, we have for sufficiently large n, by $\beta_1 < 1 + \beta_2$ and due to (3.1.21):

$$\max\left\{D_1,\, c_5 n \log(n)\,|\mathcal{C}|\,\right\} < \max\left\{n^{\beta_1}, c_5 n \log(n) n^{\beta_2}\right\} = c_5 \log(n) n^{1+\beta_2}\,. \tag{3.2.6}$$

Before we use the two inequalities in (3.2.5) and (3.2.6) to further estimate the right-hand side of (2.3.11), we need to prove another inequality, which will be used several times in the following. For $n > 2^{100}$, we have

$$\tau \leqslant \frac{1}{3}\big(1 + n^{1/100}\big) < \frac{1}{2} n^{1/100}\,.$$

From this it follows that

$$\frac{n}{\tau} > \frac{2n}{n^{1/100}} = 2n^{0.99}\,,$$

and in this way

$$\log\left(\frac{n}{\tau}\right) \;>\; \log 2 + 0.99\log n \;>\; 0.99\log n\,.$$

Finally, this gives

$$\frac{1}{\log(n/\tau)} < \frac{1}{0.99\log n} = \frac{c_6}{\log n} \qquad (n > 2^{100}) \tag{3.2.7}$$

with $c_6 := 1/0.99$, which is already mentioned for (3.2.5).

Thus, we obtain from (2.3.11) together with (2.4.48), (3.2.5), (3.2.6) and (3.2.7),

$$\begin{aligned}
|x_k| &< 2n^{1+\beta_2+\beta_2 M/2}\bigl(c_5 e\sqrt{2}\bigr)^{M/2}\bigl(\log n\bigr)^{M/2} n^{M/2}\bigl(c_5\log(n)n^{1+\beta_2}\bigr)^{M/2}\sqrt{D_1} \\
&< 2n^{1+\beta_2+(1+\varepsilon)\beta_2 c_6 n/2\tau\log n}\bigl(c_5^2 e\sqrt{2}\log^2 n\bigr)^{(1+\varepsilon)c_6 n/2\tau\log n} \\
&\quad\times n^{(1+\varepsilon)c_6 n/2\tau\log n} n^{(1+\varepsilon)c_6 n(1+\beta_2)/2\tau\log n}\sqrt{D_1} \\
&= 2n^{1+\beta_2}\bigl(c_5^2 e\sqrt{2}\log^2 n\bigr)^{(1+\varepsilon)c_6 n/2\tau\log n} n^{(1+\varepsilon)c_6 n(2+2\beta_2)/2\tau\log n}\sqrt{D_1} \\
&= 2n^{1+\beta_2}\bigl(c_5^{1/\tau} e^{1/2\tau} 2^{1/4\tau}\log^{1/\tau} n\bigr)^{(1+\varepsilon)c_6 n/\log n} \\
&\quad\times n^{\varepsilon c_6 n(1+\beta_2)/\tau\log n} n^{c_6 n(1+\beta_2)/\tau\log n}\sqrt{D_1} \\
&< 2n^{1+\beta_1/2+\beta_2}\bigl(c_5^{1/\tau} e^{1/2\tau} 2^{1/4\tau}\log^{1/\tau} n\bigr)^{(1+\varepsilon)c_6 n/\log n} e^{\varepsilon c_6 n(1+\beta_2)/\tau} e^{c_6 n(1+\beta_2)/\tau} \\
&< 2n^{1+\beta_1/2+\beta_2} e^{2\varepsilon c_6 n} e^{c_6 n(1+\beta_2)/\tau}\,,
\end{aligned} \tag{3.2.8}$$

because $\sqrt{D_1} < n^{\beta_1/2}$ holds by (3.1.21).

3.3. Conclusion of the proof attempt of Conjecture 1.

We assume that the Euler constant γ is a rational number, say, $\gamma = r/s$. We now solve (3.2.1) for x_0 and subtract a term that disappears due to (3.2.2). Then we obtain:

$$\begin{aligned}
x_0 \;&=\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \Big(\sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1}\Big)\Big) \\
&\quad - \sum_{\substack{2 \leqslant p < n/\tau \\ p \in \mathbb{P}}} \underbrace{\Big(\sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} e(k,p) x_k \Big)}_{=0} \log p \\
&=\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \Big(\sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1}\Big)\Big) \\
&\quad - \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \sum_{\substack{2 \leqslant p < n/\tau \\ p \in \mathbb{P}}} e(k,p) \log p \\
&=\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \Big(\underbrace{\sum_{m=0}^{n} \frac{t_{m+2}}{k\binom{k+m}{m}} + \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1}\Big)}_{=\mu_{n,k}} - \log k \Big) \\
&=\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \big(\gamma - R_{n,k}\big)\,, \qquad (3.3.1)
\end{aligned}$$

because, by definitions, we have $\gamma = \mu_{n,k} + R_{n,k} - \log k$. Or: we use the formulas (2.1.11) and (2.1.23). From (3.3.1), using the triangle inequality and the inequalities in (3.2.8) and (2.1.24), we obtain for all sufficiently large integers $n = 2^\alpha$:

$$\begin{aligned}
&\Big| \Big(\gamma \cdot \sum_{\substack{qn \leq k \leq 3n \\ k \notin P}} x_k \Big) - x_0 \Big| \;\leqslant\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} |x_k| \cdot R_{n,k} \\
<\;& 2 \sum_{k=qn}^{3n} \frac{n^{1+\beta_1/2+\beta_2} \cdot e^{2\varepsilon c_6 n} \cdot e^{c_6 n(1+\beta_2)/\tau}}{2k^2\binom{n+k}{n}} \;<\; \big((3-q)n+1\big) \cdot \frac{n^{1+\beta_1/2+\beta_2} \cdot e^{2\varepsilon c_6 n} \cdot e^{c_6 n(1+\beta_2)/\tau}}{(qn)^2\binom{n+qn}{n}} \\
<\;& \frac{2n}{q^2} \cdot \frac{n^{2\tau} \cdot e^{2\varepsilon c_6 n} \cdot e^{c_6 n(1+\beta_2)/\tau}}{n^2\binom{77n/32}{n}} \;\leqslant\; \frac{2}{q^2} \cdot \frac{n^{2\tau-1} 2\sqrt{n} \cdot e^{(c_6(1+\beta_2)/\tau + 2\varepsilon c_6)n}}{(5.121)^n} \\
=\;& \frac{4096}{2025\sqrt{n}} \cdot n^{2\tau} \cdot \frac{e^{(c_6(1+\beta_2)/\tau+2\varepsilon c_6)n}}{(5.121)^n} \;<\; \frac{4096}{2025\sqrt{n}} \cdot n^{2\tau} \cdot \frac{e^{c_6(1+2\varepsilon)n}}{(5.121)^n} \\
=\;& \frac{4096}{2025\sqrt{n}} \cdot n^{2\tau} \cdot \frac{\exp\big(25\,001n/24\,750\big)}{(5.121)^n}\,; \qquad (3.3.2)
\end{aligned}$$

note that $(3-q)n+1 < 2n$ holds for $n \geqslant 3$. Moreover, we used $1+\beta_1/2+\beta_2 < 1+\beta_1+\beta_2 < 2\tau$, $1+\beta_2 < \tau$ and $c_6(1+2\varepsilon) = 25\,001/24\,750$. In addition, formula (2.2.8) from Lemma 6 was applied to estimate the binomial coefficient. Now, we introduce the number x_1 by

$$x_1 \;:=\; \sum_{\substack{qn \leqslant k \leqslant 3n \\ k \notin P}} x_k \tag{3.3.3}$$

where we assume without loss of generality that x_1 is positive[13]. Due to (3.2.3), $rx_1 - sx_0 \in \mathbb{Z} \setminus \{0\}$, and with $\gamma = r/s$, it follows that

$$\bigl|\gamma x_1 - x_0\bigr| \;\geqslant\; \frac{1}{s}\,. \tag{3.3.4}$$

Now, for $n > 2^{100}$, it is obvious that

$$\tau \;<\; \frac{1}{3}\bigl(1+n^{1/100}\bigr) \;<\; \frac{n^{1/100}}{2}\,.$$

Using this, we have the following estimate:

$$n^{2\tau} \;<\; n^{n^{1/100}} \;=\; \exp\bigl(n^{1/100}\log n\bigr) \;\overset{(n\geqslant 1587)}{<}\; \exp\Bigl(\frac{n}{200}\Bigr)\,. \tag{3.3.5}$$

Combining (3.3.2) with (3.3.3), (3.3.4) and (3.3.5), we obtain the following inequalities in (3.3.7) for all $n = 2^\alpha$ with

$$\alpha \;\geqslant\; \max\Bigl\{27\,411\,206\,, \frac{\log\log s - \log\log(50/27)}{\log 2}\Bigr\} \tag{3.3.6}$$

(see Section 4.4 for a lower bound of α):

$$\begin{aligned} \frac{1}{s} \;&\leqslant\; \bigl|\gamma x_1 - x_0\bigr| \;<\; \frac{4096}{2025\sqrt{n}}\cdot\Bigl(\frac{\exp\bigl(25\,001/24\,750+1/200\bigr)}{5.121}\Bigr)^{n} \\ &<\; \frac{4096}{2025\sqrt{n}}\cdot 0.54^n \;<\; 0.54^n\,. \end{aligned} \tag{3.3.7}$$

But this is impossible by the second lower bound for α on the right-hand side in (3.3.6), because it is equivalent with

$$0.54^n \;=\; (0.54)^{2^\alpha} \;\leqslant\; \frac{1}{s}\,.$$

Assuming the truth of the third fundamental lemma, the contradiction proves the irrationality of Euler's constant.

We easily find an *upper bound* for x_1: With (3.2.8), (3.3.3), the triangle inequality, and with

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[13] Otherwise, we multiply the sum in (3.3.3) by -1.

$1+\beta_1/2+\beta_2<2\tau$ and $1+\beta_2<\tau$, we have

$$\begin{aligned}
x_1 \quad &\leqslant \quad \sum_{\substack{qn\,\leqslant\, k\,\leqslant\, 3n\\ k\,\notin\, P}} |x_k| \\
&< \quad \sum_{\substack{qn\,\leqslant\, k\,\leqslant\, 3n\\ k\,\notin\, P}} 2n^{1+\beta_1/2+\beta_2}e^{2\varepsilon c_6 n}e^{c_6 n(1+\beta_2)/\tau} \\
&\overset{(2.4.47)}{<} \quad \frac{51n}{32}\,2n^{1+\beta_1/2+\beta_2}e^{2\varepsilon c_6 n}e^{c_6 n(1+\beta_2)/\tau} \;<\; \frac{51n}{16}\,n^{2\tau}e^{2\varepsilon c_6 n}e^{c_6 n} \\
&\overset{(3.3.5)}{<} \quad \frac{51n}{16}\,\exp\Big(\frac{n}{200}\Big)\exp\big(c_6(1+2\varepsilon)n\big) \\
&< \quad \exp\Big(\frac{n}{199}\Big)\exp\big(c_6(1+2\varepsilon)n\big) \\
&= \quad \exp\Big(\Big(\frac{25\,001}{24\,750}+\frac{1}{199}\Big)n\Big) \;<\; (2.76)^n\,.
\end{aligned} \tag{3.3.8}$$

In the last estimate in (3.3.8), the inequality

$$\frac{51n}{16}\,\exp\Big(\frac{n}{200}\Big) \;<\; \exp\Big(\frac{n}{199}\Big)$$

was applied, which is equivalent to

$$39\,800\log\Big(\frac{51n}{16}\Big) \;<\; n\,.$$

With the help of a computer, we find that this inequality holds for $n \geqslant 573\,894$.

The numbers x_0 and x_1 in (3.3.7) depend[14] on n. We now write a_n and b_n instead of x_0 and x_1, respectively. Using (3.3.7) and (3.3.8), we conclude the statement in Conjecture 1:
We calculate:

$$\delta \;:=\; \frac{\log 2-3\log 3+2\log 5}{\log 3-2\log 5+\log 23} \;=\; \frac{\log(50/27)}{\log(69/25)} \;=\; \frac{\log(1/0.54)}{\log(2.76)}\,.$$

Therefore, we have the equation

$$0.54 \;=\; \frac{1}{(2.76)^{\delta}}\,,$$

and in this way, we finally obtain

$$0 \;<\; \big|\gamma b_n - a_n\big| \;\overset{(3.3.7)}{<}\; 0.54^n \;=\; \frac{1}{(2.76)^{\delta n}} \;\overset{(3.3.8)}{<}\; \frac{1}{b_n^{\delta}}\,.$$

This implies the statement in Conjecture 1. ☐

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[14] But they do not depend on r and s. In a proof of contradiction for the irrationality of a number ξ, the constructed sequence of approximations may depend on the hypothetically assumed rational number $\xi = r/s$, but this is not necessarily the case. An example of such a proof, in which the constructed sequence of approximations does not depend on r and s, is the proof of the irrationality of the number e using the partial sums of the exponential series. The well-known proof of the irrationality of π^2 due to Niven, on the other hand, uses a fictive numerator and denominator of π^2 for the construction, see Theorem 49 in [8].

4. Appendix.

4.1. Overview: The essential steps.

1.) The system (3.2.1) - (3.2.3) is transformed with a regular matrix $\mathcal{C}$. Then the transformed system is solved (first fundamental lemma). By transforming the obtained solution again using the matrix $\mathcal{C}$ (see (2.3.41)), a solution of the original system (3.2.1) - (3.2.3) is found.
The special feature of the transformed system results from the inequalities in (3.1.21): While in the system (3.2.1) - (3.2.3), the coefficients in the first equation after multiplication by V_n are large (see (2.4.34)), the first row in the transformed system $\mathcal{AC}$ has only coefficients smaller than n^{β_1} due to (3.1.21). The remaining rows are also "small".

2.) Because $A_2, \ldots, A_M$ are significantly smaller than A_1 (see (2.3.6) and (2.3.7)), only the size of the coefficients from (3.2.1) are included in the calculations, i.e., the size D_1 after the transformation with the matrix $\mathcal{C}$.

3.) The transformation matrix $\mathcal{C}$ is formed using Lemma 16 and a basis of the $(N+1)$-dimensional solution space of the equation (2.5.5): if the basis vectors are written side by side in columns, the matrix $\mathcal{C}'$ in (3.1.3) is obtained; if its last row is removed, the desired matrix $\mathcal{C}$ in (3.1.6) results.

4.) The two required inequalities in

$$\left.\begin{array}{rcl} \max\limits_{1\leqslant\mu\leqslant N+1} \left\{|c'_\mu|\right\} & < & n^{\beta_1} \\[2ex] \max\limits_{\substack{0\,\leqslant\,\nu\,\leqslant\,N\\ 1\,\leqslant\,\mu\,\leqslant\,N+1}} \left\{|c_{\nu,\mu}|\right\} & \leqslant & n^{\beta_2} \end{array}\right\} \tag{4.1.1}$$

imply the two inequalities in (3.1.21). The condition on the basis of

$$l_0x_0 + l_1x_1 + \cdots + l_Nx_N - x_{N+1} \;=\; 0$$

to be related to a balanced lattice in (2.5.6) and (2.5.7) in turn implies (4.1.1).

4.2. An algebraic remark involving $\mathbb{Z}$ modules.

The solution set of a homogeneous linear system of equations with integer coefficients can be interpreted in algebraic terms as a $\mathbb{Z}$-module over the ring $(\mathbb{Z},+,\cdot)$. We are interested here in a special solution of such a system, in which, however, a further linear equation must not disappear. This is the system in (2.4.43) to (2.4.45). In the following, we want to describe algebraically the structure from which this solution is taken. To this end, we consider a homogeneous system

$$L_\nu\bigl(X_1,\ldots,X_k\bigr) \qquad (1\leqslant\nu\leqslant n)$$

of linear forms in the unknown variables $X_1,\ldots,X_k$ with integer coefficients. In addition, let

$$L_{n+1}\bigl(X_1,\ldots,X_k,Y\bigr) \;:=\; a_1X_1+\cdots+a_nX_n - Y$$

be another linear form in the unknown variables $X_1, \ldots, X_k, Y$ with integer coefficients $a_1, \ldots, a_n$ and -1. The unknown Y does not occur explicitly in the forms $L_1, \ldots, L_n$. The two sets, given by

$$\begin{aligned} \mathcal{M}_1 &:= \left\{ \vec{X} := (X_1, \ldots, X_k, Y) \in \mathbb{Z}^{k+1} : L_\nu(\vec{X}) = 0 \ (1 \leqslant \nu \leqslant n+1) \right\}, \\ \mathcal{M}_2 &:= \left\{ \vec{X} := (X_1, \ldots, X_k, 0) \in \mathbb{Z}^{k+1} : L_\nu(\vec{X}) = 0 \ (1 \leqslant \nu \leqslant n+1) \right\}, \end{aligned}$$

are $\mathbb{Z}$-modules over the ring $(\mathbb{Z}, +, \cdot)$. We are now interested in a solution $\vec{X}$ from the set $\mathcal{M}_1 \setminus \mathcal{M}_2$. Due to $\vec{X} \in \mathcal{M}_1$, we have on the one hand

$$L_\nu\bigl(\vec{X}\bigr) = 0 \quad (1 \leqslant \nu \leqslant n) \qquad \text{and} \qquad L_{n+1}\bigl(\vec{X}\bigr) = a_1 X_1 + \cdots + a_n X_n - Y = 0\,,$$

On the other hand, due to $\vec{X} \notin \mathcal{M}_2$, $Y \neq 0$ is also true. This implies

$$L_\nu\bigl(\vec{X}\bigr) = 0 \quad (1 \leqslant \nu \leqslant n) \qquad \text{and} \qquad a_1 X_1 + \cdots + a_n X_n = Y \neq 0\,.$$

4.3. **An explicit lower bound for the main parameter n.**

During the argumentation, the parameter $n = 2^\alpha$ was chosen to be increasingly larger. The final result for the proof of Theorem 1 is

$$\alpha \geqslant 27\,411\,206\,, \tag{4.3.1}$$

apart from the additional condition

$$\alpha \geqslant \frac{\log\log s - \log\log(50/27)}{\log 2} \tag{4.3.2}$$

in (3.3.6), where s is the fictitious denominator of $\gamma = r/s$ in the proof of $\gamma \notin \mathbb{Q}$ by contradiction. In some cases, a minimum size of n can be traced back to the application of an explicit form of the prime number theorem, while other requirements are based on inequalities without the prime number theorem. In this section, the corresponding estimates with the required minimum size of n are subsequently proved. The bound in (4.3.1) is given below in 4.).

1.) J.B. Rosser and L. Schoenfeld [13] have given an explicit form of the prime number theorem:

$$\frac{m}{\log(m) - 1/2} < \pi(m) < \frac{m}{\log(m) - 3/2} \qquad (m \geqslant 67)\,. \tag{4.3.3}$$

In the following, let $m \in \mathbb{N}$ with $m \geqslant 67$ and $\delta := \varepsilon/2 = 1/100\,000$. Due to the right-hand inequality in (4.3.3), the inequality

$$\pi(m) < (1+\delta)\frac{m}{\log(m)} \tag{4.3.4}$$

is obtained from the inequality

$$\frac{m}{\log(m) - 3/2} < (1+\delta)\frac{m}{\log(m)}\,.$$

The latter one is equivalent to

$$\frac{3}{2} < \delta\Bigl(\log(m) - \frac{3}{2} \Bigr), \qquad \text{or} \qquad m > \exp\Bigl(\frac{3}{2}(1 + 1/\delta) \Bigr)\,.$$

Hence, assuming $m > \exp(150\,002)$, the inequality in (4.3.4) is fulfilled for $\delta = \varepsilon/2$. Since $n = 2^\alpha$, we obtain $m > 2^\alpha$ for

$$\alpha \geqslant 216\,408\,. \tag{4.3.5}$$

For the special application of the prime number theorem with

$$m \;=\; \frac{n}{\tau} \;=\; \frac{3n}{1+\lfloor n^{0.01}\rfloor}\,,$$

we have first due to $1+\lfloor x\rfloor < 1+x < 2x$ for real numbers $x>1$,

$$\frac{n}{\tau} \;>\; \frac{3n}{2n^{0.01}} \;=\; \frac{3}{2}n^{0.99} \;\overset{!}{>}\; \exp\big(150\,002\big)\,,$$

from one obtains

$$n \;>\; \exp\Big(\,\frac{150\,002-\log(3/2)}{0.99}\,\Big) \;=\; \exp\big(151\,516.762\ldots\big)\,.$$

Due to $n=2^\alpha$, this is fulfilled for

$$\alpha \;\geqslant\; 218\,593\,, \tag{4.3.6}$$

because

$$218\,593 \;>\; \frac{151\,517}{\log 2}\,.$$

2.) For the proof of (2.4.46), the inequality

$$\Big(1+\frac{\varepsilon}{2}\Big)\big(\gamma+1+\log(n^\beta)\big) \;<\; (1+\varepsilon)\log(n^\beta) \tag{4.3.7}$$

was used. It is equivalent to

$$\Big(1+\frac{\varepsilon}{2}\Big)(\gamma+1) \;<\; \frac{\varepsilon}{2}\,\log(n^\beta)$$

or

$$n \;>\; \exp\Big(\,\frac{(2+\varepsilon)(\gamma+1)}{\varepsilon\beta}\,\Big) \;=\; \exp\Big(\frac{100\,001(\gamma+1)}{0.01}\Big)\,.$$

This is true for $n=2^\alpha$ with $\alpha \geqslant 22\,754\,640$.

3.) The inequality in (3.2.4) is:

$$\frac{\log(c_5^{1/\tau}e^{1/2\tau}2^{1/4\tau}\,)}{\log n} + \frac{\log\log n}{\tau\log n} \;<\; \frac{\varepsilon}{1+\varepsilon}\,.$$

Due to $\tau\geqslant 7/3$, $c_5 = 384/25$ and $\varepsilon = 1/50\,000$, it follows from

$$\frac{\log\big((384/25)^{3/7}e^{3/14}2^{3/28}\big)}{\log n} + \frac{3\log\log n}{7\log n} \;<\; \frac{1}{50\,001}\,.$$

With the help of a calculator, we can see that this inequality holds for $n=2^\alpha$ with $\alpha \geqslant 500\,000$.

4.) The final inequality in (3.1.1) is equivalent with

$$\frac{\log(2n)}{2} + \log\big(\gamma+\log(3n)\big) + n\log(28\,410) \;<\; \Big(\,\frac{2\varepsilon}{3}-\frac{51\varepsilon}{80}\,\Big)n\log n \;=\; \frac{7n\log n}{12\,000\,000}\,. \tag{4.3.8}$$

For every integer $n\geqslant 1$ we have

$$\frac{\log(2n)}{2} + \log\big(\gamma+\log(3n)\big) + n\log(28\,410) \;<\; \frac{133n}{12}\,,$$

so that (4.3.8) will follow from

$$\frac{133n}{12} \;<\; \frac{7n\log n}{12\,000\,000}\,,$$

or

$$\log n \;>\; 19\,000\,000\,. \tag{4.3.9}$$

Now, we apply the above mentioned lower bound for $n = 2^\alpha$: For $\alpha > 27\,411\,206$ we have

$$\log n \;=\; \alpha \log 2 \;>\; 19\,000\,000\,,$$

so that (4.3.9) and, finally, (4.3.8) are satisfied

5.) In (2.4.46), it is claimed that $|P| > 1$. We justify this here for $n \geqslant 2^\alpha$ with $\alpha \geqslant 259$ by making use of the upper and lower bounds in (4.3.3). Then we can assume $Q \geqslant 6$, which implies $\tau \geqslant 7/3 > 4/3$:

$$\begin{aligned} |P| \;&=\; \Big|\Big\{\,\nu p \Big|\, p \in \mathbb{P} \,\wedge\, 1 \leqslant \nu \leqslant \min\{Q, \lfloor 3n/p \rfloor\} \,\wedge\, \nu \equiv 1 \pmod 2 \,\wedge\, n/\tau < p < 3n \Big\}\Big| \\ &\geqslant\; \Big|\Big\{p \,:\, p \in \mathbb{P} \,\wedge\, 3n/4 < p < 3n\Big\}\Big| \\ &=\; \Big|\Big\{p \,:\, p \in \mathbb{P} \,\wedge\, 3n/4 < p \leqslant 3n\Big\}\Big| \qquad (3n \notin \mathbb{P}) \\ &=\; \pi(3n) - \pi\Big(\frac{3n}{4}\Big) \qquad\qquad\qquad\qquad (3n/4 \in \mathbb{N}) \\ &>\; \frac{3n}{\log(3n) - 1/2} - \frac{3n}{4\log(3n/4) - 3/2} \;>\; 1\,. \end{aligned} \tag{4.3.10}$$

Using a calculator, the last inequality can be easily verified for the numbers $n = 2^\alpha$ with $\alpha \geqslant 259$.

4.4. **Tabular overview of the parameters.**

The following table summarizes the most important parameters and, where applicable, their properties:

$$\begin{aligned}
n &= 2^\alpha \quad \text{mit} \quad \alpha \geqslant 27\,411\,206\\
\beta &= \frac{1}{100} = 0.01\\
Q &= \lfloor n^\beta \rfloor \geqslant 6\\
\frac{1}{2}n^{1/100} > \tau &= \frac{1+Q}{3} = \frac{1}{3}\Big(1+\lfloor n^{0.01}\rfloor\Big) \geqslant \frac{7}{3}\\
q &= \frac{45}{32}\\
\varepsilon &= \frac{1}{50\,000}\\
P &= \Big|\Big\{\, \nu p \big| \, p \in \mathbb{P} \wedge 1 \leqslant \nu \leqslant \min\{Q, \lfloor 3n/p \rfloor\}\\
&\qquad \wedge \nu \equiv 1 \pmod 2 \wedge n/\tau < p < 3n \Big\}\Big|\\
\frac{25n}{16} < N &= \sum_{\substack{qn \le k \le 3n\\ k \notin P}} 1 < \frac{51n}{32}\\
M &= 1+\pi(n/\tau) < \frac{(1+\varepsilon)n}{\tau \log(n/\tau)}\\
c_1 &= 5.121\\
c_2 &= 28\,409\\
c_3 &= 2\\
c_4 &= \frac{96}{25}\\
c_5 &= 2c_3c_4 = \frac{384}{25}\\
c_6 &= \frac{1}{0.99}\\
\beta_1 &= \tau - 2\varepsilon\\
\beta_2 &= \tau - 1 - \varepsilon \qquad (1 < \beta_1 < 1+\beta_2 < \tau)\\
\beta_3 &= 2\\
\big|\vec{a}_1\mathcal{C}\big| &< n^{\beta_1}\\
\big|\mathcal{C}\big| &\leqslant n^{\beta_2}
\end{aligned}$$

$|.|$ always denotes the maximum norm of a matrix (or of a vector, respectively).

4.5. **A large solution basis and three examples.**

Finally, we show that under the conditions (2.5.8) to (2.5.10), a basis $\vec{b}_1, \dots, \vec{b}_{N+1}$ of the $(N+1)$-dimensional solution space already exists for the equation in (2.5.5) satisfying the inequalities

$$0.45 + \log N \;<\; \frac{|\vec{b}_\nu|}{|\vec{b}_1|} \;<\; 1.23 + \log N \tag{4.5.1}$$

and

$$1 + \frac{0.499}{N \log N} \;<\; \frac{|\vec{b}_{\nu+1}|}{|\vec{b}_\nu|} \;<\; 1 + \frac{1.13(1+\psi)}{N \log N} \tag{4.5.2}$$

for $\nu = 2, 3, \dots, N$. So, it's about the equation

$$l_0 x_0 + l_1 x_1 + \cdots + l_N x_N - x_{N+1} \;=\; 0\,, \tag{4.5.3}$$

where $l_0 = -V_n$ and $l_\nu = V_n \mu_{n,k}$ for $\nu = 1, 2, \dots, N$. Every $(N+2)$-dimensional vector

$$\vec{b}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ l_0 \end{pmatrix}, \quad \vec{b}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ l_1 \end{pmatrix}, \quad \dots \quad, \quad \vec{b}_N := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ l_{N-1} \end{pmatrix}, \quad \vec{b}_{N+1} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ l_N \end{pmatrix}$$

is obviously a solution of (4.5.3). Introducing the unit vectors

$$\vec{e}_j \;:=\; \big(\, \underbrace{0, \dots, 0}_{j-1}, 1, \underbrace{0, \dots, 0}_{N+2-j} \,\big)^T \;\in\; \mathbb{Z}^{N+2} \qquad (1 \leqslant j \leqslant N+2)\,,$$

we obtain

$$\vec{b}_{i+1} \;:=\; \vec{e}_{i+1} + l_i \vec{e}_{N+2} \qquad (0 \leqslant i \leqslant N)\,.$$

On the one hand, the vectors $\vec{b}_1, \dots, \vec{b}_{N+1}$ are linearly independent over $\mathbb{Q}$. Let $\beta_0, \beta_1, \dots, \beta_N$ be rational numbers satisfying

$$\vec{0} \;=\; \sum_{i=0}^{N} \beta_i \vec{b}_{i+1} \;=\; \sum_{i=0}^{N} \big(\, \beta_i \vec{e}_{i+1} + \beta_i l_i \vec{e}_{N+2} \,\big) \;=\; \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \\ \beta_0 l_0 + \cdots + \beta_N l_N \end{pmatrix}. \tag{4.5.4}$$

In this way, the zero vector can only be represented for $\beta_0 = \beta_1 = \ldots = \beta_N = 0$. On the other hand, if $x_0, \dots, x_{N+1}$ is any solution of (4.5.3), then set $\beta_i := x_i$ for $i = 0, \dots, N$ on the right-hand side of (4.5.4). Due to (4.5.3), the lower component of the vector on the right-hand side of (4.5.4) equals to x_{N+1}, and in this way the vector $(x_0, \dots, x_{N+1})^T$ is a linear combination of the vectors $\vec{b}_1, \dots, \vec{b}_{N+1}$. Thus, $\{\vec{b}_1, \dots, \vec{b}_{N+1}\}$ forms an integer basis of the (N+1)-dimensional solution space of (4.5.3) over $\mathbb{Q}$. Obviously, we now have $|\vec{b}_\nu| = l_{\nu-1}$, so that the desired relations in (4.5.1) and (4.5.2) follow from (2.5.9) and (2.5.10) for suitable chosen subscripts ν with $1 \leqslant \nu \leqslant N+1$. $\square$

But the basis $\{\vec{b}_1, \ldots, \vec{b}_{N+1}\}$ must still be reduced, because it does not satisfy the inequality (2.5.6) guaranteed in the third fundamental lemma.

Examples. Here are three examples: for $N = 25, 31, 38$, omitting the fact that N should be very large. These numbers correspond to $n = 20$, $n = 25$ and $n = 30$, respectively. Because they are below the minimum value $2^{27\,411\,206}$ required in the above text, the left-hand side of the inequality $25n/16 < N < 51n/32$ is not satisfied here. However, the underlying Diophantine equations (4.5.3) with their coefficients $l_0, l_1, \ldots, l_N$ are taken from specific applications. For all three examples, we show that both inequalities in (2.5.6) and (2.5.7) are satisfied.

In each of the three examples, we start with the coefficients computed by the formulas in (2.5.4). Then, we build the standard basis $\vec{b}_1, \ldots, \vec{b}_{N+1}$ and reduce it using the LLL algorithm with the maximum reduction parameter 1 [4, pp. 68-69]. Although the LLL algorithm is designed for the Euclidean vector norm, in all three cases we find a basis $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{N+1}$ that satisfies the right-hand inequality in (2.5.6) with $\varepsilon = 0$ (based on the maximum norm!) and is thus a basis guaranteed by Bombieri and Vaaler [3]. We do not show all the intermediate steps of the calculations, but we do provide all the necessary data to verify the inequalities in the third fundamental lemma.

Case 1. $N = 25$.
The Diophantine equation in (4.5.3) has the following coefficients $l_0, l_1, \ldots, l_{25}$:

$$\begin{aligned}
l_0 &= 21873408844214959845873870450690398880630183768883200000000,\\
l_1 &= 8627991261839090218886375379068473224483169704679625697420,\\
l_2 &= 8702145511864636110218618339803767427101053823731429600016,\\
l_3 &= 8843313257768946451609081168470312333595611148419213767922,\\
l_4 &= 8910621364852199856636992435018834267026181830325912275872,\\
l_5 &= 8975919971670625761088675691002063952523823972869889122752,\\
l_6 &= 9039325596212166151643696808454795790991154230504320655328,\\
l_7 &= 9100944907151428834577977806427598270977818876917139679224,\\
l_8 &= 9219208350742966552169746324373200797141382591449000449664,\\
l_9 &= 9276025594145553263602044421017429882919280557100539608624,\\
l_{10} &= 9331404270653523492192646228993021871165617610707895250120,\\
l_{11} &= 9438124991498958873849309515426313471276351112690570003776,\\
l_{12} &= 9539880123641087236674785544263940108605438109275130818941,\\
l_{13} &= 9589035920036005874481698666706959091630581949392125022562,\\
l_{14} &= 9637111281296869274464948670996039564095749766251952382088,\\
l_{15} &= 9730203665940316214253609163829712115053579038725261640943,\\
l_{16} &= 9775305075846488913032186262652389951526611822548728079688,\\
l_{17} &= 9819495283538100532012477351983075187563623705164836999024,\\
l_{18} &= 9862810379841995523215397035226337855091938874667410944920,
\end{aligned}$$

$$\begin{aligned}
l_{19} &= 99052843529344406432015110700339772663111379264443425833\!70, \\
l_{20} &= 9987835315322798596529630876510367289915335604493556769920, \\
l_{21} &= 10027971136525664276482228337449273402369711056993749325684, \\
l_{22} &= 10067383750287846253431046626348181086027716007772809847158, \\
l_{23} &= 10106098758914336314059173111026755780656495066098450912488, \\
l_{24} &= 10144140428799347364616693674910025045169169006095240678288, \\
l_{25} &= 10218294678824893254058925258338881159718689223720223218617.
\end{aligned}$$

The LLL algorithm calculates a reduced basis $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{26}$ of the solution lattice Λ of the Diophantine equation (4.5.3). We arrange the basis vectors according to the length of their maximum norms and obtain the following norms $T_1, T_2, \ldots, T_{26}$ in ascending order:

$$81, 89, 92, 95, 107, 108, 110, 111, 114, 115, 117, 118, 120, 123, 123, 124, 128, 129, 130, 130,$$
$$132, 133, 135, 146, 146, 157$$

The product of these norms is

$$\begin{aligned}
\prod_{m=1}^{26} T_m &= 801760136455045657126007528911274141381146705920000000 \\
&= 8.017601364 \cdots 10^{53}
\end{aligned}$$

while

$$\begin{aligned}
d(\Lambda) &= \sqrt{1 + l_0^2 + l_1^2 + \ldots + l_{25}^2} \\
&= 4.768491377 \ldots \cdot 10^{58}\,.
\end{aligned}$$

Therefore, the right-hand inequality in (2.5.6) in the third fundamental lemma is satisfied. Furthermore, we obtain

$$T_1 \,=\, 81 \,>\, 12.529\ldots \,=\, \sqrt{157} \,=\, \sqrt{T_{26}}\,,$$

so that inequality (2.5.7) is also satisfied.

Case 2. $N = 31$.
Here, we have the following coefficients $l_0, l_1, \ldots, l_{31}$:

$$\begin{aligned}
l_0 &= 425138595789891532924267633879606577330056095394334374618988544 00000000, \\
l_1 &= 1768888866725879602897913968692358824123771859443878871981578768107636 32, \\
l_2 &= 17805372281382980605070141650049993060648645153617327940637267152553542 4, \\
l_3 &= 179187492925498599837375135453840708619334499257945204928386356420008576, \\
l_4 &= 180291811106942765471774836778078904585743764562159290124263263704595993, \\
l_5 &= 181368168840434675657861791511841117100014886339910300600344788793121565, \\
l_6 &= 183442427028769735787729778046108901500114581296188028091805418099547384,
\end{aligned}$$

$$\begin{aligned}
l_7 &= 18542017244108395065302166737116619524151613920257256945744870208287 2638,\\
l_8 &= 1863755802791172343809598088050307572074305250446374801261829170186 28280,\\
l_9 &= 1873099884330589462988384332797951298132202029813769370965749335191 88672,\\
l_{10} &= 1891193619041955584841151182464969940170793734983142817378139709536 31871,\\
l_{11} &= 1899959673849515112856530434721025276293164077428340418326114300085 80128,\\
l_{12} &= 1908548624469639497490157636588842645763973018610259238916826369685 87088,\\
l_{13} &= 1916967485631287917184454221725269864608653163828503278424010809761 95748,\\
l_{14} &= 1925222863385503636660852879811129640533206020190934124729436411639 00836,\\
l_{15} &= 1941267733428781172072115207064268286542202282899674989538693525572 27552,\\
l_{16} &= 1949068660476132247441740944000001966257181427248569066190386203086 06855,\\
l_{17} &= 1956729022603773339820378071687912935250675120951319351522466990355 15283,\\
l_{18} &= 1964253795957887567547841897477328811438157928856254083116060287289 90656,\\
l_{19} &= 1971647697032750713076865362947594890550592468190011339649553241371 14744,\\
l_{20} &= 1986060555107248325752673492085909427209135035962838346585572348200 81790,\\
l_{21} &= 2000000774338414001555609287817466168106351224507082258538043084417 87616,\\
l_{22} &= 2006803136990598927051341783110536050733284345990424308574407271394 59509,\\
l_{23} &= 2013498371358031566784225508756004545997094257980972964681549994181 31537,\\
l_{24} &= 2020089799450796377572374901608702577283352073065322080034952227472 63360,\\
l_{25} &= 2026580591113741075704256676436027731647249016489269640429519490590 71808,\\
l_{26} &= 2039272237947363899127528290735544784827560590324010243514182174920 09744,\\
l_{27} &= 2045478751033491032162422897122757317895724642183153137786005552253 85792,\\
l_{28} &= 2051595958669066080731900045556694148530409020016949416966132171500 05120,\\
l_{29} &= 2063572485744857154015189210122141453464873005961226228339790543936 26786,\\
l_{30} &= 2075220847157275611624218259737616704579088396130273425648995624521 28752,\\
l_{31} &= 2080927491172541066664195457043644794755221275498101944670619525935 58044.
\end{aligned}$$

From the LLL algorithm, we obtain the norms $T_1, T_2, \ldots, T_{32}$:

$$\begin{aligned}
&106, 121, 123, 127, 127, 129, 130, 131, 133, 133, 133, 134, 136, 138, 139, 139, 140, 148,\\
&149, 149, 149, 156, 157, 165, 166, 169, 179, 180, 186, 187, 193, 223
\end{aligned}$$

with the product satisfying

$$\begin{aligned}
&\prod_{m=1}^{32} T_m\\
= \;& 239502067369855521665849525063521198423562595920096077792718217216 0000\\
= \;& 2.395020673 \cdots 10^{69},
\end{aligned}$$

while

$$\begin{aligned}
d(\Lambda) &= \sqrt{1 + l_0^2 + l_1^2 + \ldots + l_{31}^2}\\
&= 1.081797802 \ldots \cdot 10^{72}.
\end{aligned}$$

Moreover, we have

$$T_1 = 106 > 14.933\ldots = \sqrt{223} = \sqrt{T_{32}}\,,$$

Again, the conditions (2.5.6) and (2.5.7) are fulfilled.

Case 3. $N = 38$.

The coefficients $l_0, l_1, \ldots, l_{38}$ are:

$$\begin{aligned}
l_0 &= 58101956964632826468282669155404602415569263221696033908921314020368404547895296000000000,\\
l_1 &= 252070446978820156841433804255815592434198266286496998290106873971828204594014336701462880,\\
l_2 &= 253406182977349245056912504106984218993708552974921120543859157655918716201484895530132605,\\
l_3 &= 254711899880937950483059225396861485425788643030978158773521275089523892835151713099991680,\\
l_4 &= 255988917373241908926831009719003675574426925877007978812307632178570472644964758495260421,\\
l_5 &= 258461714258634156707486183407270259352903747773769417445460564471917676163733298771748309,\\
l_6 &= 259659735196328909627228211525718548781826701059454022760142669803195593670937247758442496,\\
l_7 &= 260833552027460497309786972259062321341652078354754962242235859031614118184393345252005555,\\
l_8 &= 261984123426409794304329742747700964799591395606543727834383291338550916515703831445579680,\\
l_9 &= 263112352215214655826394535210282145642846256319145394343081293171350281899297118388253955,\\
l_{10} &= 265305139127505161897458890348410568604111906267762846859733228310827456821553517789930315,\\
l_{11} &= 266371259992742797073612958854217585774644963702008798490120793320909736341065849325815200,\\
l_{12} &= 267418170424026412287920902146437522559188547480921161230739052843172003695165038264041275,\\
l_{13} &= 268446550490948005139348687101522018428942423262831715459457351132614569501829630792655360,\\
l_{14} &= 269457044777532995144735507767287721573182155469341340463883068733848064696980866788292855,\\
l_{15} &= 271426791274027708724186635807472996401431764558401780641706196707450713248960840177298375,\\
l_{16} &= 273331944148567059540440647386712311021777168421842288139566465650813091174188119695436095,\\
l_{17} &= 274261595292897417477893608803479990372341986316559749160597918307734960306164114330935840,\\
l_{18} &= 275176605651723914948613592597336191685123014881925664872980155973708883190806092703689775,\\
l_{19} &= 276077429230608207843094987346469683453977398912671049569129999177877265417244510648524800,\\
l_{20} &= 276964499239310008488012622304059496356799601959075593842535131657903335086445635148600010,\\
l_{21} &= 278699014819499552545457151698757639512291057305094396261594202940011301726375368151676970,\\
l_{22} &= 279547233635202672357931126139722376956771025299667571765727555454799508920869492004875520,\\
l_{23} &= 280383247439419964304621354248471369123830556885073287442120549558817629196842221672716890,\\
l_{24} &= 282020030520594920138586299248823666511099484321694245645727703314206811863333800924091002,\\
l_{25} &= 283611964539747831127225149555285337899787090231105925644604072759359206943140394137228018,\\
l_{26} &= 284391868289421260740887087904512637421559794357954419241471042937371646065660151065135040,\\
l_{27} &= 285161441884037763380476095990409929667699419790112084455814732404975928426859558865891410,\\
l_{28} &= 285920955404702264068447340744155312918610193950667712496946267324007545498127784994812160,\\
l_{29} &= 286670668477175419257494650786949683236741384630942699597758597381682522174617767210911650,\\
l_{30} &= 288141682667117466965314044693561945652646081536174829836360918887334538952787022637587650,\\
l_{31} &= 288863455389465925328559005900143601299639310280624730542333387676942832762797014032998400,\\
l_{32} &= 289576371781187582616837668567780020744163663774965955634228582282972075092193165355469670,\\
l_{33} &= 290976486685987175719021017688847583763500126999673745670342153970138708710869541982539270,\\
l_{34} &= 291664091834893104562157603794837308351037969078989683739718413922769122379170443984864000,\\
l_{35} &= 292343654633870678513661333362018615587514956128618506515752807518044806718507020156075030,\\
l_{36} &= 293015361039493758575835623306109142695562141372452451132228028601128093290668133535941440,\\
l_{37} &= 293679390632399766729140031189230759219286994946332297090481658244568705394113580434199030,\\
l_{38} &= 294985107535988472155286751344874741404071472926929457601241514648448046927112865358102350.
\end{aligned}$$

From the LLL algorithm, we obtain the norms $T_1, T_2, \ldots, T_{39}$:

$$130, 134, 136, 146, 147, 150, 156, 157, 161, 163, 165, 166, 168, 171, 171, 172, 172, 174,$$
$$177, 178, 180, 181, 183, 186, 187, 191, 196, 197, 197, 202, 202, 215, 215, 224, 230, 231,$$
$$246, 247, 254$$

with the product satisfying

$$\begin{aligned} & \prod_{m=1}^{39} T_m \\ = \; & 11165831421167960282328866142867369763511176334234246088357410916489339483036057 6 \cdot 10^8 \\ = \; & 1.116583142 \cdots 10^{88}\,, \end{aligned}$$

while

$$\begin{aligned} d(\Lambda) &= \sqrt{1 + l_0^2 + l_1^2 + \ldots + l_{38}^2} \\ &= 1.701944337 \ldots \cdot 10^{90}\,. \end{aligned}$$

Moreover, we have

$$T_1 = 130 > 15.937\ldots = \sqrt{254} = \sqrt{T_{39}}\,,$$

The conditions (2.5.6) and (2.5.7) are fulfilled.

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