A Conjecture on Euler's Constant

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In these lines, I want to touch on a passion project of Carstens, and how it shaped his own path. Maybe after reading this text you can be inspired to contribute your own thoughts to Carsten in the same way that he did with me. We are looking to open a door for you to another adventure in mathematics -a research project- whose story begins several decades ago.

1 The Irrationality of Euler's Constant γ

The beginning of this particular story takes place in Germany, over four decades ago. This is a time when the headlines in the Federal Republic of Germany were dominated by strikes demanding 35-hour workweeks, the Flick affair, forest die back, and the rise of public television. The year was 1984, when Germany was a divided country, still shaped by the walls of the Cold War. But behind the walls of the University of Hannover, Carsten Elsner was only a student attending his very first lecture in number theory by a gentleman he knew as Professor Rieger.

Carsten was curious and very pensive when it came to solving mathematical problems. He was inquisitive, and while Prof. Rieger gave his lecture on the topic of Euler's constant γ in the context of the asymptotic expansion of the harmonic series, Carsten was drawn towards the mysteries of this number in the same way I had been drawn into the mysteries of the Zopf number $\mathfrak{z} := [1, 2, 3, \ldots]$. But Euler's constant doesn't have a regular continued fraction expansion with such an elegant pattern of partial quotients:

$$\gamma = [0, 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, 1, 11, 3 \ldots].$$

Instead, we define Euler's constant by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right), \tag{1}$$

and it's still not known whether γ is even rational or irrational. Since Carsten had decided to specialize in number theory during his studies, his path towards Diophantine approximation -and thus continued fractions- had already begun. Meanwhile, the irrationality of Euler's constant had never left his mind.

Keep in mind that there was no social media yet. Meaningful discussions in mathematics were "analog" and usually took place in more private settings. Carsten, can you tell me about some of the discussions with your professors on this topic and how they reacted to your goals of solving this problem?

Carsten: I would like to answer your question within a broader, historical context. Two points are essential here:

1.) In 1984, I was a student who was focused solely on studying and had no interest whatsoever in research, let alone in tackling mathematical problems that had remained unsolved for centuries. Nevertheless, these problems, which we young people encountered in lectures, were naturally fascinating!

2.) At that time, only a few years had passed since Roger Apéry had proven the irrationality of $\zeta(3)$ in 1978. My mentor, Prof. Rieger, had discussed Apéry's proof with him personally, and he drew my attention to Alf van der Poorten's article in *The Mathematical Intelligencer* from 1979 *(A proof that Euler missed)*. This is how I became more familiar with the problem of irrationality proofs. In seminars at the time, I presented the transcendence proofs for the numbers π and e and became fascinated by Ivan M. Niven's elegant proof of the irrationality of these two numbers.

In seminars, I met Paul Erdösz, who encouraged me to prove the irrationality of the number

$$\sum_{n=0}^{\infty} \frac{1}{n!+1},$$

and I also met Alan Baker in person, who gave a presentation at a conference on applications of his linear form theorems to logarithms. But I didn't dare discuss a possible approach to proving the irrationality of γ with either P. Erdösz or A. Baker: at the time, I felt these people were way above my level as a student. Later, shortly before completing my studies in 1987, I studied in detail Klaus Friedrich Roth's proof of the approximation of real-algebraic numbers by rational numbers and discussed the essential details of this proof with my mentor, Prof. Rieger. Thus, I became increasingly familiar with the question of the algebraic character of given numbers, and the old problem of the irrationality of γ occasionally came to the forefront again when my studies were not taking up so much of my time. By that time, I had already begun to consider a possible approach to proving the irrationality of γ and had discussed it with Prof. Rieger. He handed me a copy of Paul Appell's paper *Sur la nature arithmétique de la constante d'Euler* from 1926, saying: "Perhaps you can correct the proof"! In this article Paul Appell attempted to prove the irrationality of γ using a special series representation by J. Ser. Unfortunately, his proof contained a fundamental flaw, as he himself noted a few days later.

Christopher: To answer your question about specific discussions with my professors at the time regarding my possible approaches to proving the irrationality of γ, there is very little to say: After graduating in 1988, I spoke only once with my former analysis professor about the approach of a series transformation, and he encouraged me to present this approach in the Institute of Mathematics' own research series at the University of Hannover. That was my entry point into deeper considerations regarding a proof of the irrationality of γ.

Christopher: Ahh.. so Professor Rieger seemed supportive! Enough at least to challenge you to correct the erroneous proof of Paul Appell. What amazes me is that Appells proof had been read by mathematicians for decades before you saw it in the 80s. The fact that γ is still not proven to be rational or irrational, even after Appell came so close, is a testament to its difficulty. But perhaps all things happen for a reason.

At this point in time, we have progressed another 4 decades, from the time Carsten began pursuing this problem, and he has now whittled it down to a single inequality. Proving this means completing the final step in a problem that has sat on the minds of countless mathematicians for centuries. But before we get into the details of the progress, Carsten, can you describe a timeline of any critical points in your progress, and some of the feelings involved during the moments when you felt like the solution was imminent?

Carsten: To tell the truth: I was never euphoric in my belief that I would be able to reach the goal in the near future. However, it was frustrating on a few occasions when an approach didn't lead to the goal, and motivating on a few occasions when I had reached a milestone. I learned quite a bit through these milestones. Three such subgoals can be identified in the work. Here I must now dip a bit into the mathematical context:

1. Estimating the solution of a mixed system of equations and inequalities based on an idea by P.G. Becker-Landeck from a 1988 paper. In this context, proving the linear independence of a linear form from many other linear forms using subtle estimates of binomial coefficients was a first major partial success.

2. Transformation of a mixed system of equations and inequalities with some "large" coefficients into a system where a "small" solution can be guaranteed. The columns of the regular transformation matrix consists of elements from vectors that form the basis of a special lattice.

3. Basis reduction for a lattice, where the resulting reduced basis consists only of vectors whose lengths do not differ too much.

But back to your question, Christopher, regarding the extent to which I became emotionally invested in the partial successes and setbacks: simply put, not very much. When working on such a notoriously difficult problem, one shouldn't go into it with too many expectations, especially since my engagement with this topic stretched over many years. I did, after all, work on many other projects in the meantime and often let Euler's constant "rest" for long periods. Now there is still one missing piece of the puzzle (the proof of an inequality between the largest and smallest successive minima of a lattice), but I cannot estimate how difficult a proof of this inequality might be. Numerical examples for the outstanding conjecture are encouraging, but they do not meet the required condition of very large parameter values, which would exceed the computational capabilities of computers. Nevertheless, I remain calm about this. Another question also bothers me: No one has reviewed my work so far. Does it perhaps already contain fundamental errors that cannot be corrected? This uncertainty is indeed an emotional burden!

Christopher: Well, you've put 40 years of thought and work into this problem, and it sounds like you have reached a place in life where the feedback from fresh minds would be extremely useful. I can only imagine how it might feel to have the thoughts of other experts and interested mathematicians to contribute their own two cents, so that you may venture past the plateau representing your current progress. One thing I am sure of is that we will soon remove the emotional burden of not knowing whether any fundamental flaws already exist in your reasoning. I'm sure of this because today we will male a "call to arms"... a challenge to any of the mathematicians out there.

Carsten has made incredible progress on the problem concerning the rationality or irrationality of γ. He has boiled it down to a single inequality. I have taken the liberty of posting a downloadable PDF titled "Euler's Constant and Series Transformations with Bases of Balanced Lattices" on www.continuedfractions.net and on my own SubStack page. In this PDF, Carsten breaks the whole problem down, giving his progress to date, and commences to prove the irrationality of γ. Unfortunately, as I mentioned, there is a small lemma -a conjecture, really- giving a statement regarding an inequality. The proving of this lemma, along with Carsten's hard work, will establish once and for all, that γ is an irrational number. Ladies and gentlemen, YOUR MISSION, SHOULD YOU CHOOSE TO ACCEPT IT, is:

1) Read the PDF. We don't believe any fundamental flaws are present, but go ahead... don the referee gloves and see for yourself.

2) Help Carsten solve his conjecture regarding the largest and smallest successive minima of a lattice.

Here is what I would like: I am only the teller of stories in this particular journey. If you feel like you can contribute meaningfully to Carsten's paper as a collaborator (links given above), or if you would like to serve as a set of expert eyes, email Carsten directly at carsten.elsner73@gmail.com .

Before we wrap this up, Carsten, can you tell us a bit about the conjectured final piece of the puzzle, and what it would mean to you to finally put this problem to rest?

Carsten: The remaining lattice problem can be formulated quite simply in mathematical terms:

I consider an infinite sequence of lattices: For infinitely many (very large) natural numbers N, there exists a specific $(N+1)$-dimensional lattice Λ_N in the $(N+2)$-dimensional Euclidean space. This lattice is the solution lattice of a linear Diophantine equation. We consider the successive minima $\lambda_1 < \lambda_2 < \ < \lambda_{N+1}$ of this lattice Λ_N with respect to the maximum norm of the vectors. By Minkowski's second fundamental theorem, the product of these successive minima satisfies the inequality $\lambda_1\lambda_2\cdots\lambda_{N+1} \leqslant d(\Lambda_N)$, where $d(\Lambda_N)$ denotes the lattice constant of Λ_N. Each of the individual successive minima corresponds to a lattice vector, and the set of these lattice vectors forms a (small) basis of the lattice. For this small basis, I would now like to see the inequality $\sqrt{\lambda_{N+1}} < \lambda_1$ satisfied, which means that the lengths of the basis vectors do not differ too much from one another in terms of the maximum norm. In general, of course, this does not hold, but in the case of the specific lattices I am working with, this seems to be consistently satisfied. I am familiar with many analytical properties of my lattices, but I suspect that to prove the inequality $\sqrt{\lambda_{N+1}} < \lambda_1$, additional arithmetic properties of the Diophantine equation generating the lattices must be taken into account. But I am no lattice expert, and at this point I am completely at a loss. One promising fact should not remain unmentioned here: I am not necessarily restricted to the bases of the lattices that result from the successive minima. It is sufficient to consider a lattice basis $\vec{t}_1, \ldots, \vec{t}_{N+1}$ of Λ_N such that the product of the maximum norms is less than $N^{\varepsilon N} d(\Lambda_N)$, where ε is a small fixed positive quantity. In doing so, it suffices to choose N sufficiently large. Here, again the inequality $\sqrt{\|\vec{t}_{N+1}\|} < \|\vec{t}_1\|$ is required for my argument.

Christopher: You ask what would mean to me to finally put this problem to rest. In my lifetime so far, significant conjectures have been proven. I need only mention the proofs of the irrationality of $\zeta(3)$, the Bieberbach conjecture, the conjecture on Fermat's Last Theorem, and the Poincaré conjecture. These have been magnificent mathematical milestones, and if someone were to join me in completing the proof of the irrationality of Euler's constant, it would be a spectacular conclusion of my own mathematical career. In 1997, I had a monument erected at the site of Ferdinand von Lindemann's birthplace in Hannover (Germany) to remember Lindemann's magnificent contribution to proving the transcendence of the circular number π. But we must not forget that this pioneering work would not have been possible without the proof of the transcendence of the number e, which Charles Hermite had provided nine years earlier. The same applies to the great mathematical achievements mentioned above, which were always preceded by significant research results from other mathematicians. What matters, then, is not so much **who** achieves a significant result, but that **we** achieve it, in accordance with David Hilbert's statement: "**We** must know - **we** will know."

Christopher, thank you for showing such great interest in my work on Euler's constant and for being interested yourself in continuing it and possibly bringing it to a conclusion. Perhaps the transcript of this conversation will help with what mathematical journals and other publication outlets strictly reject: a call to continue and to complete a project within a math community.